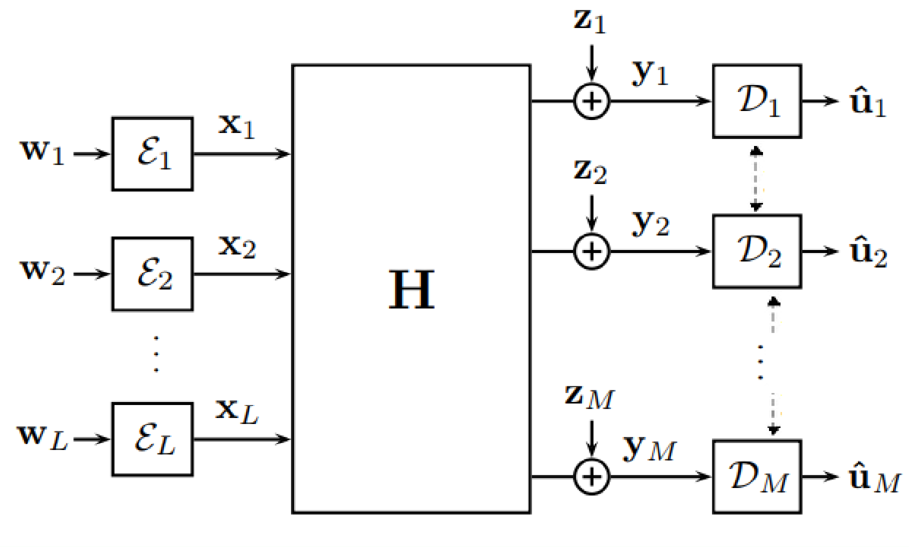
Lattice-based coding schemes and Diophantine approximations

Evgeniy Zorin

University of York

23 January 2018



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Single-user receiver

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The mutual information in this case is equal to

$$C \;=\; \log\det\left(\mathbf{I}_m + \mathrm{SNR}\cdot \mathbf{H}^t \cdot \mathbf{H}\right).$$

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1. Zero-forcing,
2. Minimum mean-squared error (MMSE).

B. Nazer and M. Gastpar, “Compute-and-forward: Harnessing interference through structured codes”, IEEE Transactions on Information Theory (2011)

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Key step: selection of an integer matrix **A** to approximate the channel matrix **H**.

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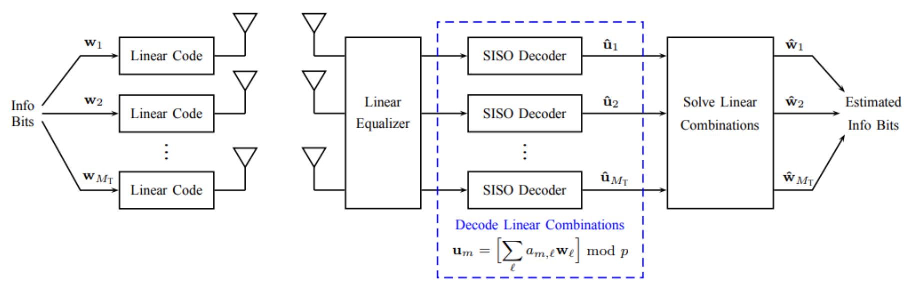

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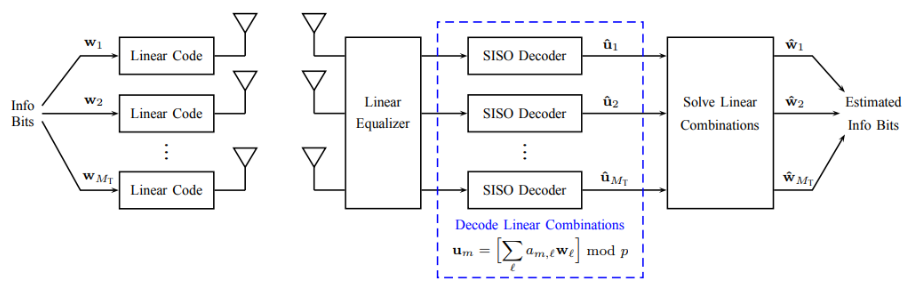

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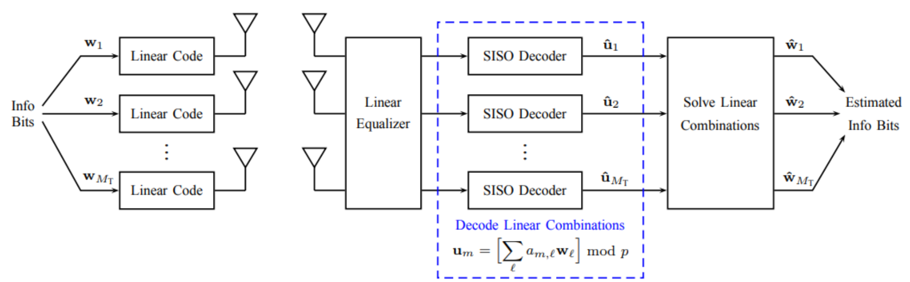

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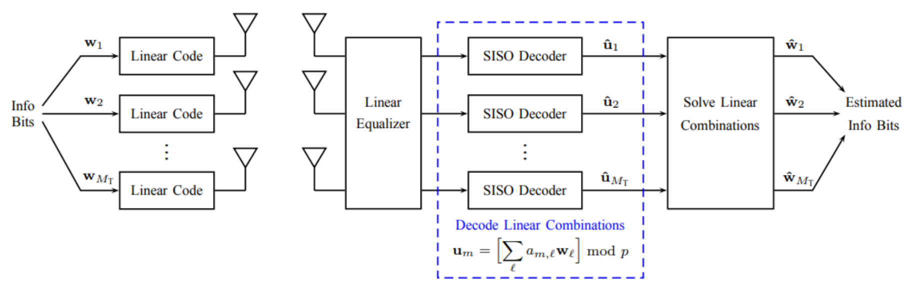

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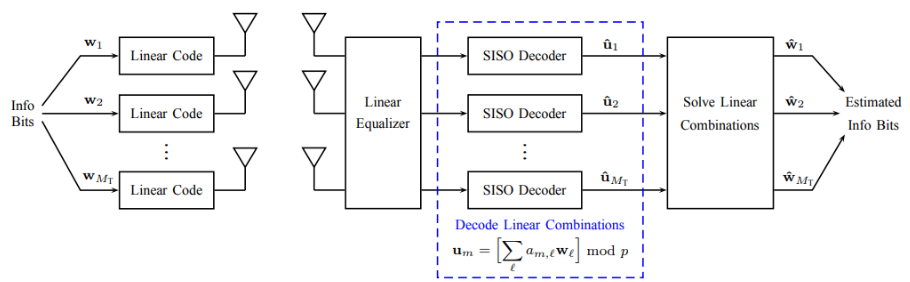

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To make the Integer-Forcing scheme to be successful, decoding over **all** sub-channels should be correct.

We define the effective SNR at the *k*th sub-channel as

$$\mathrm{SNR}_{eff,k} := \left(\mathbf{a}_k^T \left(\mathbf{I} + \mathrm{SNR}\mathbf{H}^T\mathbf{H}\right)\mathbf{a}_k\right)^{-1},$$

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Theorem (Zhan, Nazer, Ordentlich and Gastbar)

Integer-forcing can achieve any rate

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Remark

The optimal choice of the matrix $\mathbf{A}$ *is given by*

$$\mathbf{A}^{opt} := \mathrm{argmin}_{\substack{\mathbf{A}\in\mathbb{Z}^{n\times n} \\ \det \mathbf{A}\neq 0}} \max_{k=1,\dots,n} \mathbf{a}_k^T\left(\mathbf{I} + \mathrm{SNR}\mathbf{H}^T\mathbf{H}\right)\mathbf{a}_k.$$

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$$\frac{1}{n^2} M_n(\mathbf{I}_n + \mathrm{SNR} \cdot \mathbf{H}^T \cdot \mathbf{H}) < \mathrm{SNR}_{eff} \leq M_n(\mathbf{I}_n + \mathrm{SNR} \cdot \mathbf{H}^T \cdot \mathbf{H}).$$

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Problem

Assume that the channel matrix $\mathbf{H}$ *has coefficients distributed with respect to a given law (e.g. Gaussian distribution independently for each entry).*
Let $\kappa \in (0,1)$.
Find the best possible value of $s \geq 0$ *such that the event* $SNR_{eff} \geq s$ *is realised with probability greater than* κ.

Refined version of the problem

Define

$$\mathcal{H}_{m,n}(C_0, \mathrm{SNR}) := \left\{ \mathbf{H} \in \mathbb{R}^{n\times m} \ : \ \log\det\left(\mathbf{I}_m + \mathrm{SNR}\cdot\mathbf{H}^T\cdot\mathbf{H}\right) = C_0 \right\}.$$

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Problem

Assume that the channel matrix H is chosen randomly from the set $\mathcal{H}_{m,n}(C_0, SNR)$, according to any given probability distribution on this set. Let $\kappa \in (0,1)$.
Find the best possible value of $s \geq 0$ such that the event $\mathrm{SNR}_{\mathrm{eff}} \geq s$ is realised with probability greater than κ; equivalently, determine the cumulative distribution function of the quantity $\mathrm{SNR}_{\mathrm{eff}}$ seen as a random variable.

Mathematical interpretation

So, we would like to find cumulative distribution function of

$$M_m(\mathbf{I}_m + \mathrm{SNR}\mathbf{H}^T\mathbf{H})$$

when

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Problem (Mathematical version of the previous problem)

For a given probability measure μ on the set Σ_d^{++}, estimate the probability $\mu\left(M_d(\Sigma) \leq \delta\right)$ as a function of $\delta > 0$.

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Mathematical simplification

The quantity

$$M_d(\mathbf{Q}) := \min_{\mathbf{a}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mathbf{a}^T \cdot \mathbf{Q} \cdot \mathbf{a}$$

does not change if we replace the matrix $\mathbf{Q}$ by $\mathbf{A}^T\mathbf{Q}\mathbf{A}$, for any $\mathbf{A} \in \mathrm{SL}_d(\mathbb{Z})$.

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Then, there is a bijection between the locally symmetric set

$$X_d := SL_d(\mathbb{Z})\backslash SL_d(\mathbb{R})/SO_d(\mathbb{R})$$

and $\Sigma_{d,red}^{++}$, given by

$$\phi : \overline{g} \in X_d \mapsto g^T \cdot g \in \Sigma_{d,red}^{++}.$$

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Consider

$$\begin{aligned} p_{X_d}(\delta) &= (\phi_*\mu_{X_d})\left(\left\{\overline{\Sigma} \in \Sigma^{++}_{d,red} \ : \ M_d(\overline{\Sigma}) \leq \delta\right\}\right) \\ &= \mu_{X_d}\left(\{\overline{g} \in X_d \ : \ M_d(\phi(\overline{g})) \leq \delta\}\right) \end{aligned}$$

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Theorem (Kleinbock & Margulis, 1998)

The following inequalities hold for any $\delta > 0$:

$$\frac{V_d}{2\zeta(d)}\delta^{d/2} - c_d \frac{V_d^2}{4}\delta^d \le p_{X_d}(\delta) \le \frac{V_d}{2\zeta(d)}\delta^{d/2}. \tag{1}$$

Here, ζ denotes the Riemann zeta function and c_d a strictly positive constant which, when $d \geq 3$, can be taken to be

$$c_d = \frac{1}{\zeta(d)\cdot\zeta(d-1)}.$$

Final Estimate

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Theorem (Adiceam and Z.)

Let $\delta \in (0, 1)$. *Then,*

$$0 \leq 1 - \int_{\sqrt{\delta}}^{\infty} g_f \leq m_f(\delta) \leq 1 - \int_{I_d(\delta)} G_f \leq 1, \qquad (2)$$

where

$$I_d(\delta) := \left(\sqrt{\delta}, +\infty\right)^d.$$

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$$G_f(\beta) \ := \ 2^d \cdot \prod_{i=1}^{d} \beta_i^{d-i+1} \cdot \int_{\mathbb{R}^p} (f \circ \varphi_{chol})(\beta, \mathbf{u}) \cdot \mathrm{d}\lambda_p(\mathbf{u}).$$

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and, given any $\beta_1 > 0$

$$g_f(\beta_1) \ := \ \int_{(\mathbb{R}_{>0})^{d-1}} G_f(\beta_1,\ \tilde{\beta}) \cdot \mathrm{d}\lambda_{d-1}(\tilde{\beta}). \tag{4}$$

Adiceam and Z., “On the Minimum of a Positive Definite Quadratic Form over Non–Zero Lattice points. Theory and Applications”, Journal de Mathmatiques Pures et Appliques (2018)

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Theorem (Adiceam and Z.)

Let $\delta \in (0, 1)$. *Then,*

$$0 \leq 1 - \int_{\sqrt{\delta}}^{\infty} \widetilde{g}_{\widetilde{f}} \leq \widetilde{m}_{\widetilde{f}}(\delta) \leq 1 - \int_{\Delta_{d-1}(\delta)} \widetilde{G}_{\widetilde{f}} \leq 1, \tag{5}$$

$$\Delta_{d-1}(\delta) := \left\{\beta' \in (\mathbb{R}_{>0})^{d-1} \ : \ \left(\forall i = 1, \ldots, d-1, \ \beta_i > \sqrt{\delta}\right) \wedge \left(\prod_{i=1}^{d-1} \beta_i < \frac{1}{\sqrt{\delta}}\right)\right\}.$$

Interference Alignment

and

Diophantine approximations

MISO channel

$$y = h_1 x_1 + h_2 x_2.$$

MISO channel

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$$y = \begin{cases} 0 & \text{if} \quad x_1 = x_2 = 0 \\ h_1 & \text{if} \quad x_1 = 0 \text{ and } x_2 = 1 \\ h_2 & \text{if} \quad x_1 = 1 \text{ and } x_2 = 0 \\ h_1 + h_2 & \text{if} \quad x_1 = x_2 = 1\,, \end{cases}$$

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$$y = h_1 x_1 + h_2 x_2 + z.$$

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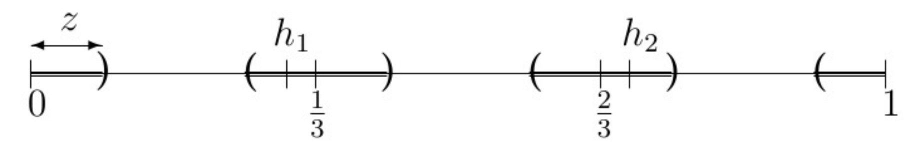

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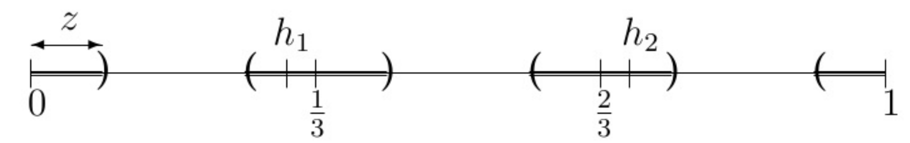
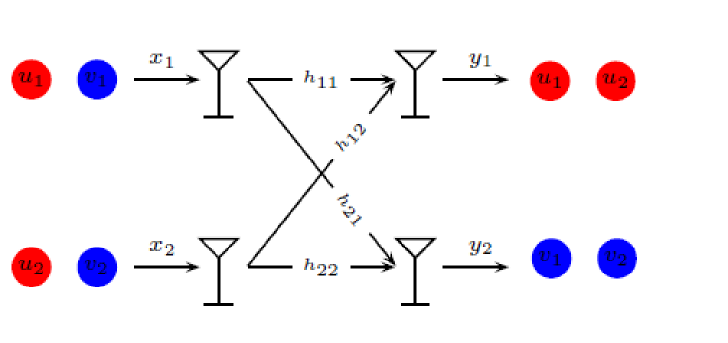
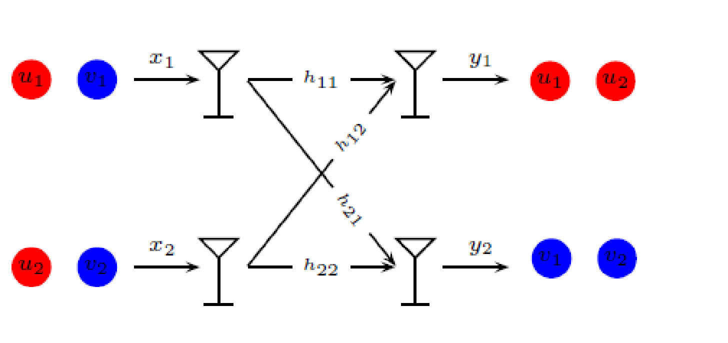


Picture become more complicated if we transmit x_1 and x_2 from bigger interval, e.g. $x_1, x_2 \in [1, \ldots, 16]$.

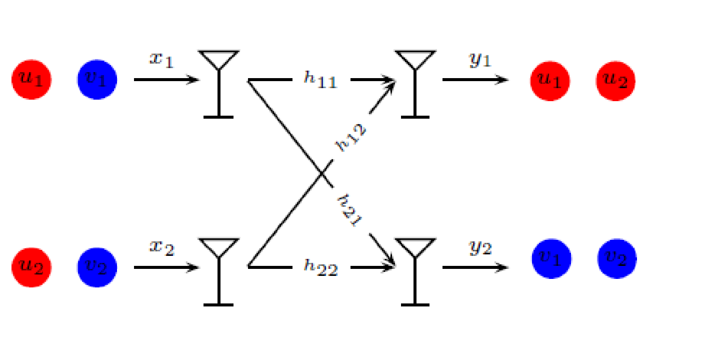


u_1
v_1
x_1
h_{11}
y_1
u_1
u_2
h_{12}
h_{21}
u_2
v_2
x_2
h_{22}
y_2
v_1
v_2

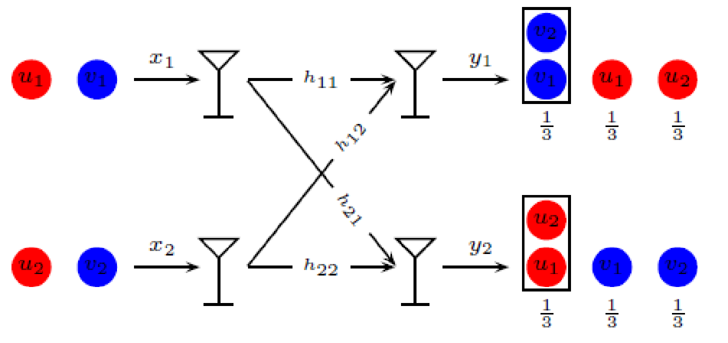


Precoding:

$$x_1 = h_{22}u_1 + h_{12}v_1$$
$$x_2 = h_{21}u_2 + h_{11}v_2$$



$$y_1 = (h_{11}h_{22})u_1 + (h_{21}h_{12})u_2 + (h_{11}h_{12})(v_1 + v_2)$$
$$y_2 = (h_{21}h_{12})v_1 + (h_{11}h_{22})v_2 + (h_{21}h_{22})(u_1 + u_2)\,.$$



$$
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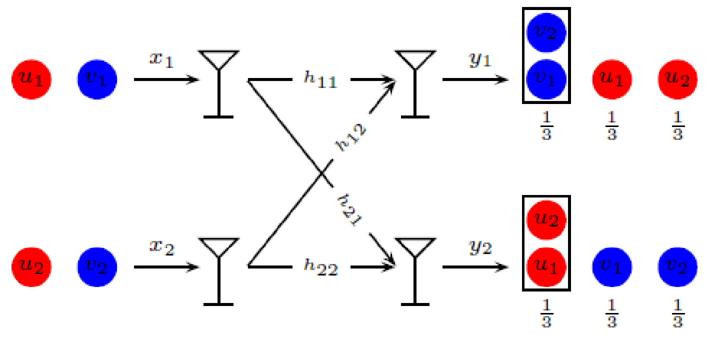

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Problem

Estimate outage probability of this scheme.

1. ...
2. A. S. Motahari, S. Oveis-Gharan, M.A. Maddah-Ali, A. K. Khandani. Real interference alignment: exploiting the potential of single antenna systems, *IEEE Trans. Inform. Theory* 60 (2014), no. 8, 4799–4810.
3. A. S. Motahari, S. Oveis–Gharan, M.A. Maddah–Ali, A. K. Khandani. “Real Interference Alignment”, 2010.
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Definition

Given a real positive function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(r) \to 0$ as $r \to \infty$ (i.e. a so called approximating function), let

$$\mathcal{B}(\psi) := \{x \in \mathbb{R} : |qx - p| > \psi(q) \text{ for a.b.f.m. } (q,p) \in \mathbb{N} \times \mathbb{Z}\},$$

where 'for a.b.f.m.' reads 'for all but finitely many'.

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Theorem (Khintchine, 1924)

Let ψ be an approximating function. Then

$$|\mathcal{B}(\psi)| = \begin{cases} \text{FULL} & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 0 & \textit{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty \quad \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

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Example

Let $\log^+(q) := \max(1, \log(q))$. Set $\psi(q) = \frac{1}{q \log^+(q)^2}$. Clearly,

$$\sum_{q=1}^{\infty} \frac{1}{q \log^+(q)^2} < \infty,$$

so for almost all $x \in \mathbb{R}$ we have

$$\|qx\| > \frac{1}{q \log^+(q)^2} \quad \text{for a.b.f.m. } q \in \mathbb{N}.$$

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However if we are interested in just one value $q = 1$, the inequality (6) fails for **all** $x \in \mathbb{R}$.
Moreover, the inequality

$$\|qx\| > \frac{10^{-100}}{q \log^+(q)^2} \tag{7}$$

fails for all integers $q \in [-1024; 1024]$ on a set of x of strictly positive Lebesgue measure. Indeed, it fails on, say,

$$x \in [2 - \frac{10^{-100}}{1024 \log(1024)^2}; 2 + \frac{10^{-100}}{1024 \log(1024)^2}]$$

Mutidimensional case

Definition

Let $m \in \mathbb{N}$ and $\psi : (0, \infty) \to (0, \infty)$, define

$$\mathcal{B}_m(\psi) := \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{a}.\mathbf{x}\| > \psi(|\mathbf{a}|) \text{ for a.b.f.m. } \mathbf{a} \in \mathbb{Z}^m,\ \mathbf{a} \neq \mathbf{0}\},$$

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Theorem

Let $m \in \mathbb{N}$, ψ be an approximating function and

$$S_m(\psi) := \sum_{q=1}^{\infty} q^{m-1}\psi(q)\,. \tag{8}$$

Then for any $m > 1$

$$|\mathcal{B}(\psi)|_m = \begin{cases} \mathrm{FULL} & \textit{if} \quad S_m(\psi) < \infty\,, \\ 0 & \textit{if} \quad S_m(\psi) = \infty\,. \end{cases}$$

Corollary

Suppose that ψ *is an approximating function and* $S_m(\psi) < \infty$. *Then for almost every* $\mathbf{x} \in \mathbb{R}^m$ *there exists a constant* $\kappa(\mathbf{x}) > 0$ *such that*

$$\|\mathbf{a}.\mathbf{x}\| \; > \; \kappa(\mathbf{x})\,\psi(|\mathbf{a}|) \qquad \textit{for all} \;\; \mathbf{a} \in \mathbb{Z}^m,\; \mathbf{a} \neq \mathbf{0}\,. \tag{9}$$

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Theorem

Let $m \in \mathbb{N}$ *and* μ *be a probability measure on* $\mathbb{R}^m$ *absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^m$*. Let* $\Psi : \mathbb{Z}^m \to \mathbb{R}^+$ *be any function,* $M_\Psi := \max_{\mathbf{a}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \Psi(\mathbf{a})$ *and*

$$\Sigma(\Psi) := \sum_{\mathbf{a}\in\mathbb{Z}^m} \Psi(q).$$

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Then for any $\delta \in (0,1)$ there is an (explicit) constant $\kappa > 0$ depending on μ, $\Sigma(\Psi)$ and δ only such that

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$$\kappa := \frac{1}{2}\min\left\{\frac{1}{M_\Psi},\ \left(\frac{\delta}{2\Sigma(\Psi)\Sigma_f}\right)^{1/m}\right\}.$$

Example

Assume that x follows the standard normal distribution, $\mathcal{N}(0,1)$, and that $\psi(n) = \frac{1}{n \cdot \log(n)^2}$, $n \geq 2$, with $\psi(1) = 1$.

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Let δ takes the values 0.5, 0.25, 0.1 and 0.01. Then we have from previous calculations:

δ	0.5	0.25	0.1	0.01
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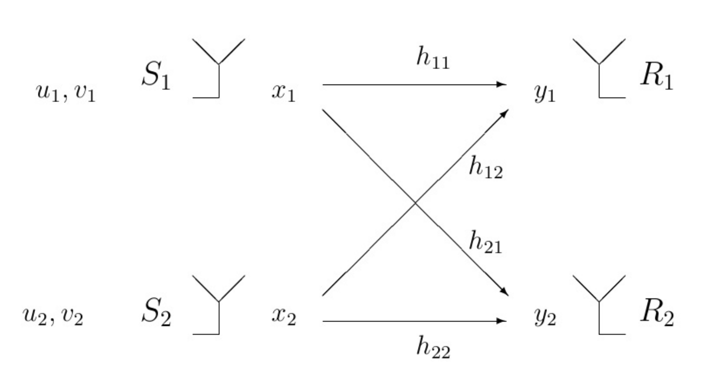
In particular, see that for 99% of the values of the random variable x that has normal distribution $\mathcal{N}(0,1)$ the inequality

$$\|nx\| > \frac{3 \cdot 10^{-6}}{n \cdot \log(n)^2}$$

holds for all $n \in \mathbb{N}$.

MANIFOLDS

MANIFOLDS



Let $\mathcal{M}$ be a manifold parametrized by $\mathbf{f} : \mathcal{U} \to \mathbb{R}^m$, where $\mathcal{U} \subset \mathbb{R}^d$ is an open set (might be all $\mathbb{R}^d$). Let $l \in \mathbb{N}$ and assume that at every point $\mathbf{x} \in \mathcal{U}$ among all partial derivatives of the order up to l, $\frac{\partial^k \mathbf{f}}{\partial x_1^{a_1} \ldots \partial x_d^{a_d}}(\mathbf{x})$, $a_1, \ldots, a_d \in \mathbb{N} \cup \{0\}$, $k = a_1 + \cdots + a_d$, $1, \leq k \leq l$, there are d linearly independent vectors. Then we say that manifold $\mathcal{M}$ is **non-degenerate**.

Example

Consider the simplest case, when $\mathcal{M}$ is a planar curve, defined by a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. Then the condition that the curve $\mathcal{M}$ is non-degenerate means that at every point $\mathbf{x}$ at least one of derivatives of $\mathbf{f}$ of an order up to l is non-zero.

Let $\Psi : \mathbb{Z}^m \setminus \{0\} \to (0, \infty)$ be a function satisfying

$$\Sigma(\Psi) := \sum_{\mathbf{q} \in \mathbb{Z}^m \setminus \{0\}} \Psi(\mathbf{q}) < \infty \tag{10}$$

$$\Psi(\mathbf{q}) \leq \frac{C_\Psi}{\prod_i \max(1, |q_i|)}, \tag{11}$$

where $C_\Psi > 0$.

$$\mathcal{B}_m(\Psi, \kappa, \mathcal{M}) := \{\mathbf{x} \in \mathcal{U} : \|\mathbf{a}.\mathbf{f}(\mathbf{x})\| > \kappa\Psi(\mathbf{a}) \ \text{ for all } \ \mathbf{a} \in \mathbb{Z}^m, \ \mathbf{a} \neq \mathbf{0}\}.$$

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Theorem (Adiceam, Beresnevich, Leveseley, Velani and Z.)

Let $l \in \mathbb{N}$ and let $\mathcal{M}$ be a compact d–dimensional C^{l+1} submanifold of $\mathbb{R}^n$ and assume it is l–non–degenerate at every point. Let μ be a probability measure supported on $\mathcal{M}$ absolutely continuous with respect to $|\,.\,|_\mathcal{M}$. Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ be a monotonically decreasing function in each variable satisfying (10). Then there exist positive constants κ_0, C_0, C_1 depending on Ψ and $\mathcal{M}$ only such that for any $0 < \delta < 1$, the inequality

$$\mu(\mathcal{B}_n(\Psi, \kappa) \cap \mathcal{M}) \geq 1 - \delta$$

holds with

$$\kappa := \min\left\{\kappa_0,\ C_0 \Sigma_\Psi^{-1} \delta,\ C_1 \delta^{d(n+1)(2l-1)}\right\}.$$

One more example

So, admitting a small error $\delta > 0$ we can always find a constant κ such that for m-tuple of integers $\mathbf{a} \in \mathbb{Z}^m \setminus \{0\}$ we have

$$\|\mathbf{a}.\mathbf{f}(\mathbf{x})\| > \kappa\Psi(\mathbf{a}).$$

For example for any $\delta > 0$, we can provide a constant κ such that with probability $> 1 - \delta$ we have

$$\|\mathbf{a}.\mathbf{f}(\mathbf{x})\| > \frac{\kappa}{\prod_i \max(1, a_i \log(a_i)^2)}$$

Remark

If we transmit only such m-tuples $\mathbf{a} = (a_1, \ldots, a_m)$ that $|a_i| \leq Q$, $i = 1, \ldots, m$, we can optimize this result.

One more example

Define

$$\Psi(a_1, \ldots, a_m) := \begin{cases} \frac{1}{Q^m}, & \text{if } |a_i| \leq Q \text{ for all } i = 1, \ldots, m. \\ 0 & \text{otherwise.} \end{cases}$$

We have $S_\Psi = 1$. Then we have that for any $\delta > 0$ there exists an explicite constant κ such that for any $\mathbf{a} = (a_1, \ldots, a_m)$ such that $|a_i| \leq Q$, $i = 1, \ldots, m$ we have

$$\|\mathbf{a}.\mathbf{f}(\mathbf{x})\| > \frac{\kappa}{Q^m}.$$

Thank you!