Algebraically Structured LWE, Revisited

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London-ish Lattice Coding & Crypto
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'Algebraic' Learning With Errors

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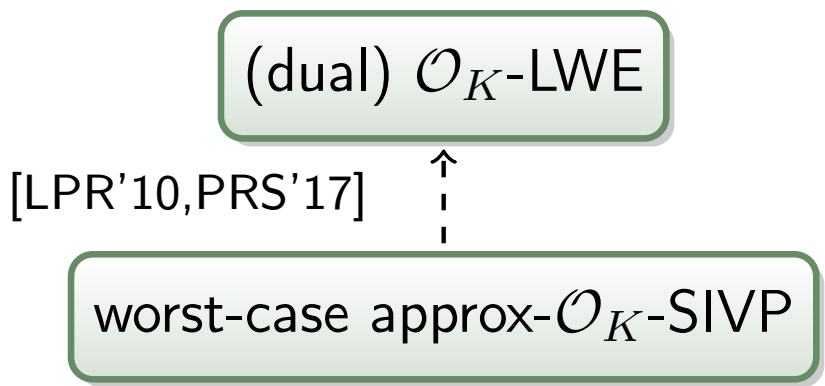
- But these reductions are often difficult to understand and use:
  - ⋆ Several steps between problems of interest
  - ⋆ Complex analysis and parameters
  - ⋆ Frequently large blowup and distortion of error distributions, across different metrics
  - ⋆ Sometimes non-uniform advice that appears hard to compute

Prior Hardness of Ring-LWE and MP-LWE



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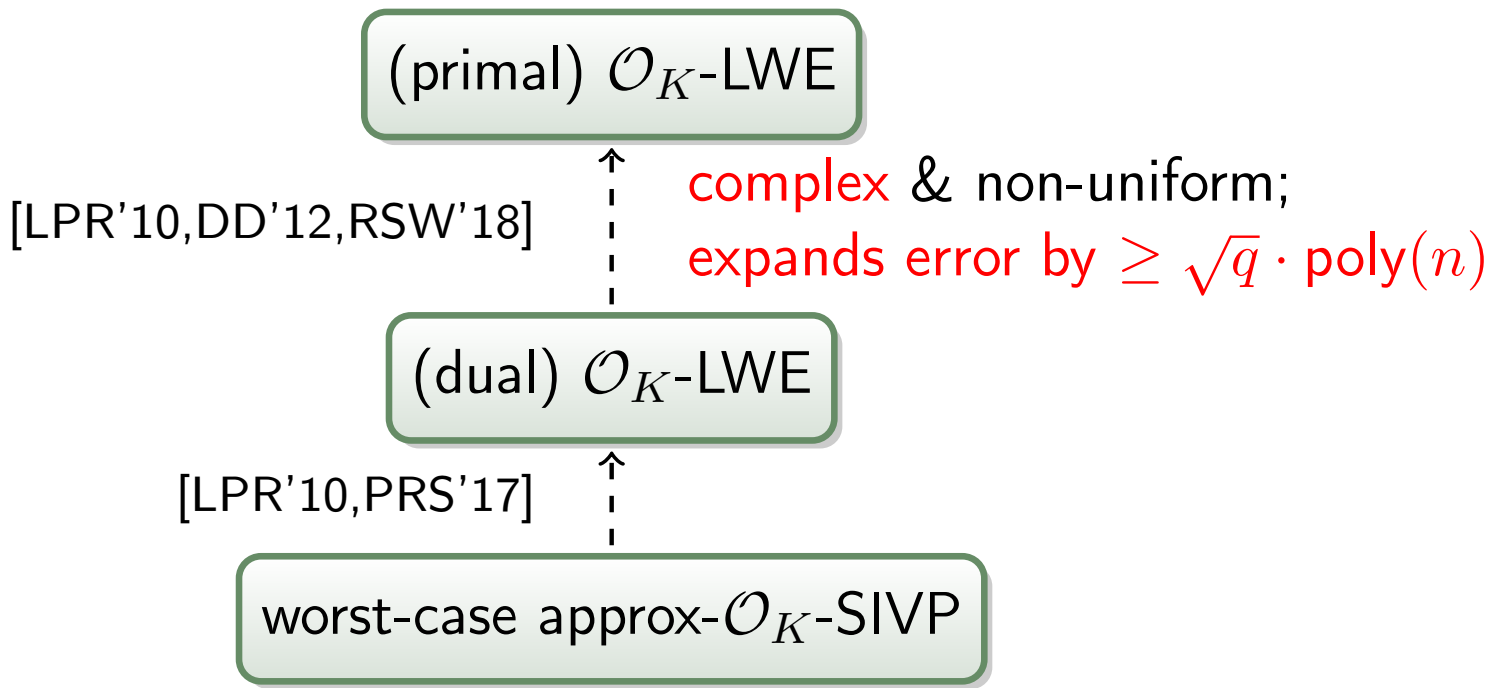

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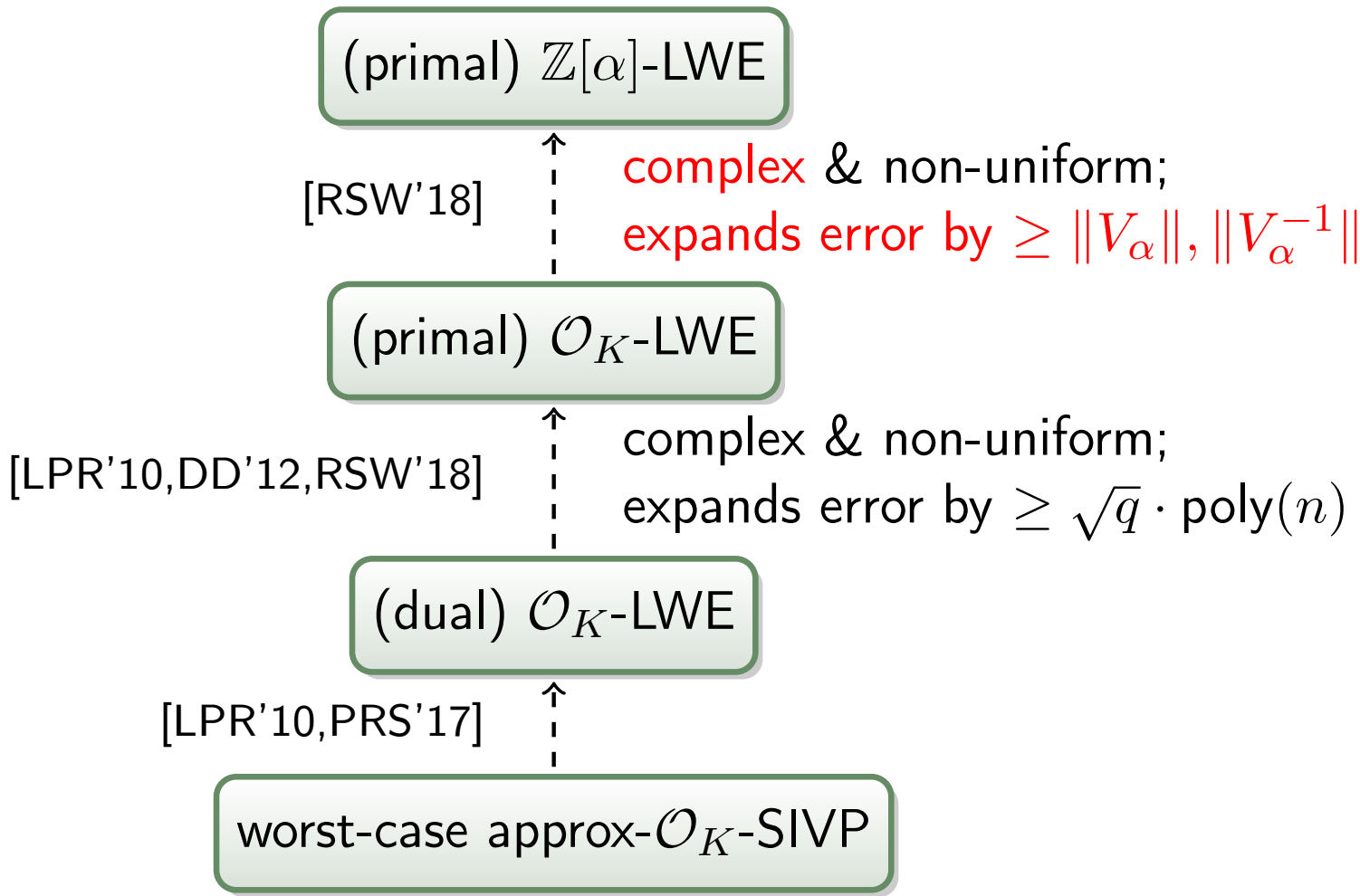

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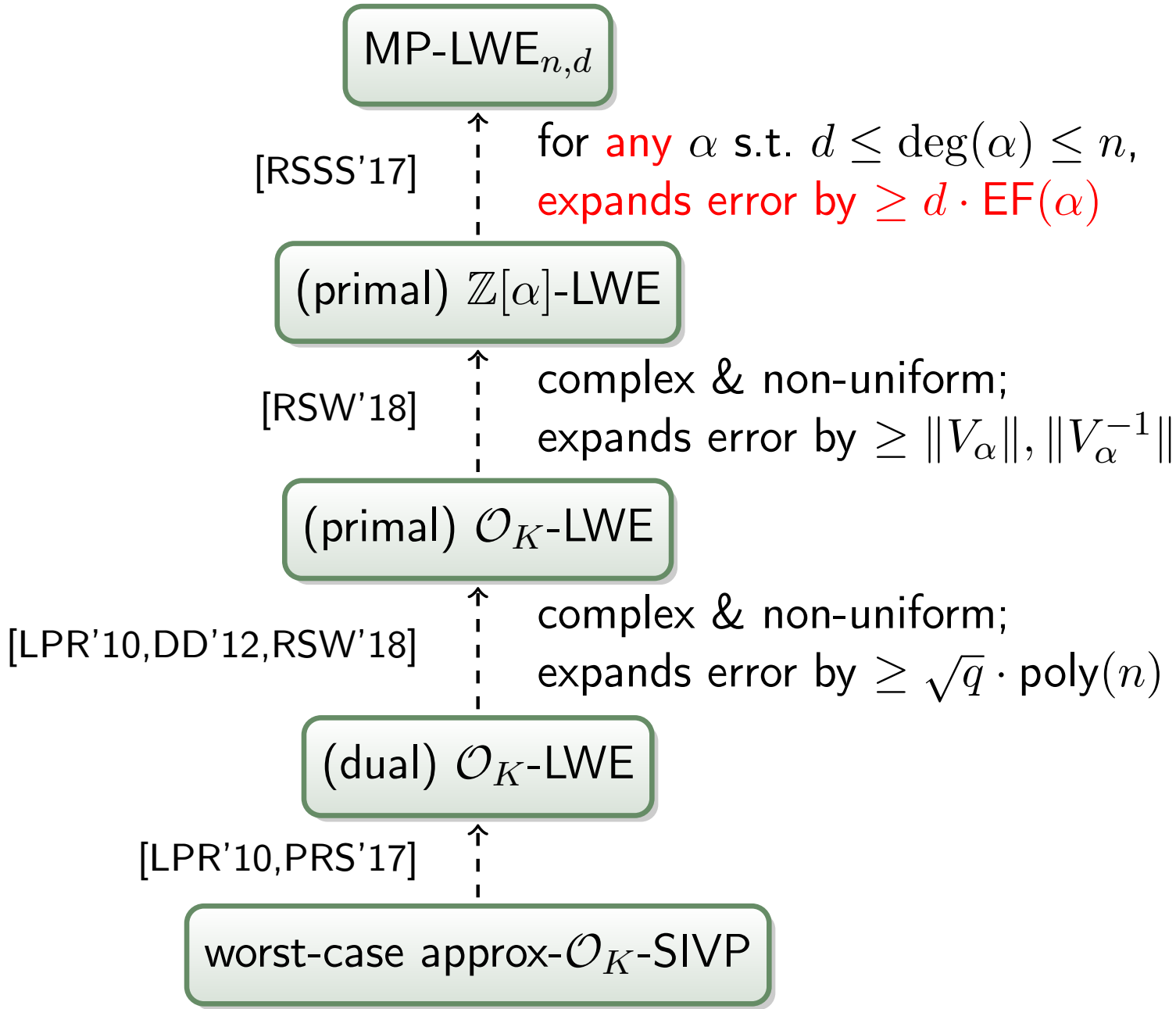

Our Contributions

Definitions

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2. For any order $\mathcal{L} = \mathbb{Z}[\alpha]$ with $d \leq \deg(\alpha) \leq n$,

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with error expansion $\|V_\alpha\|$.

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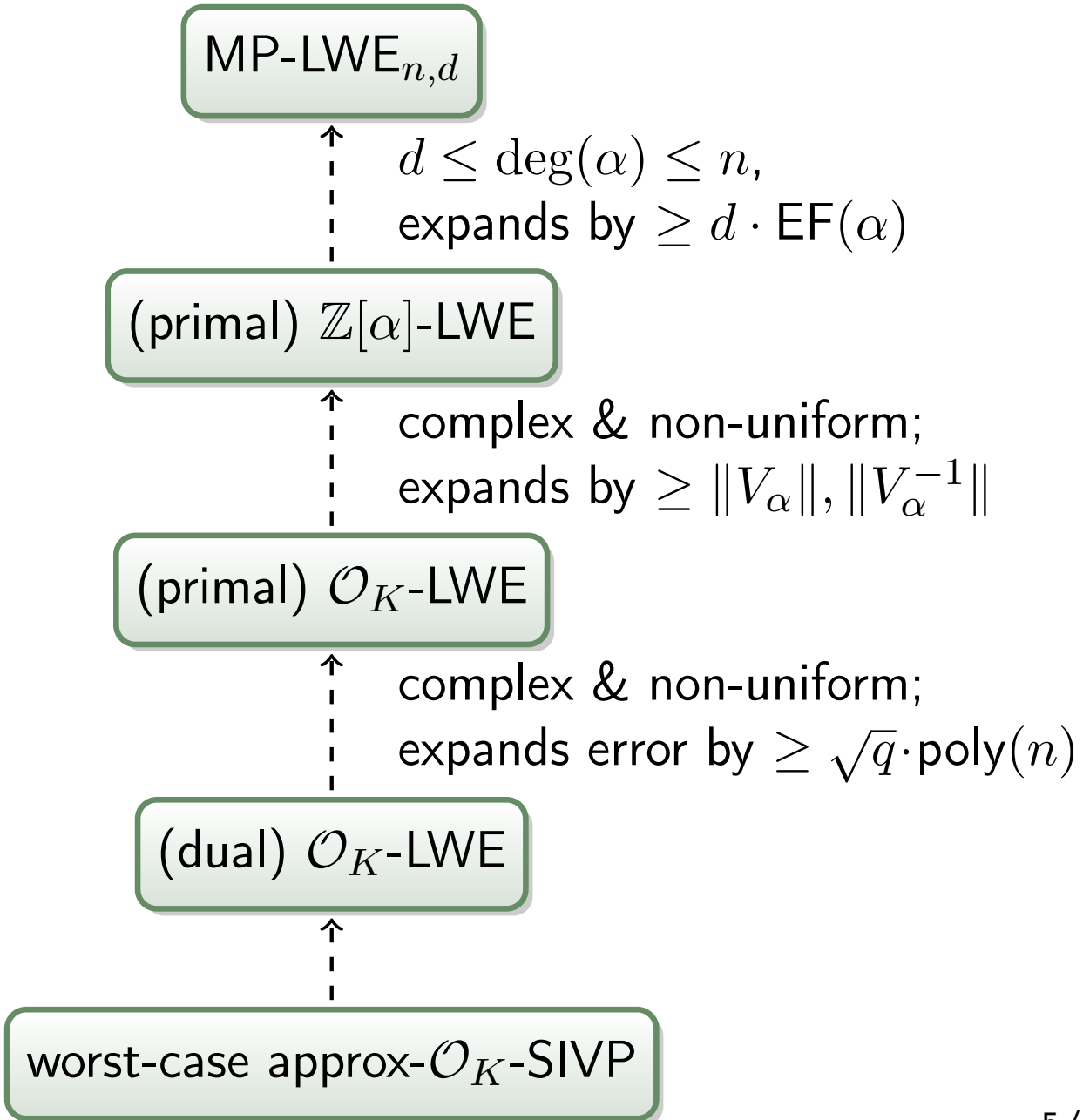

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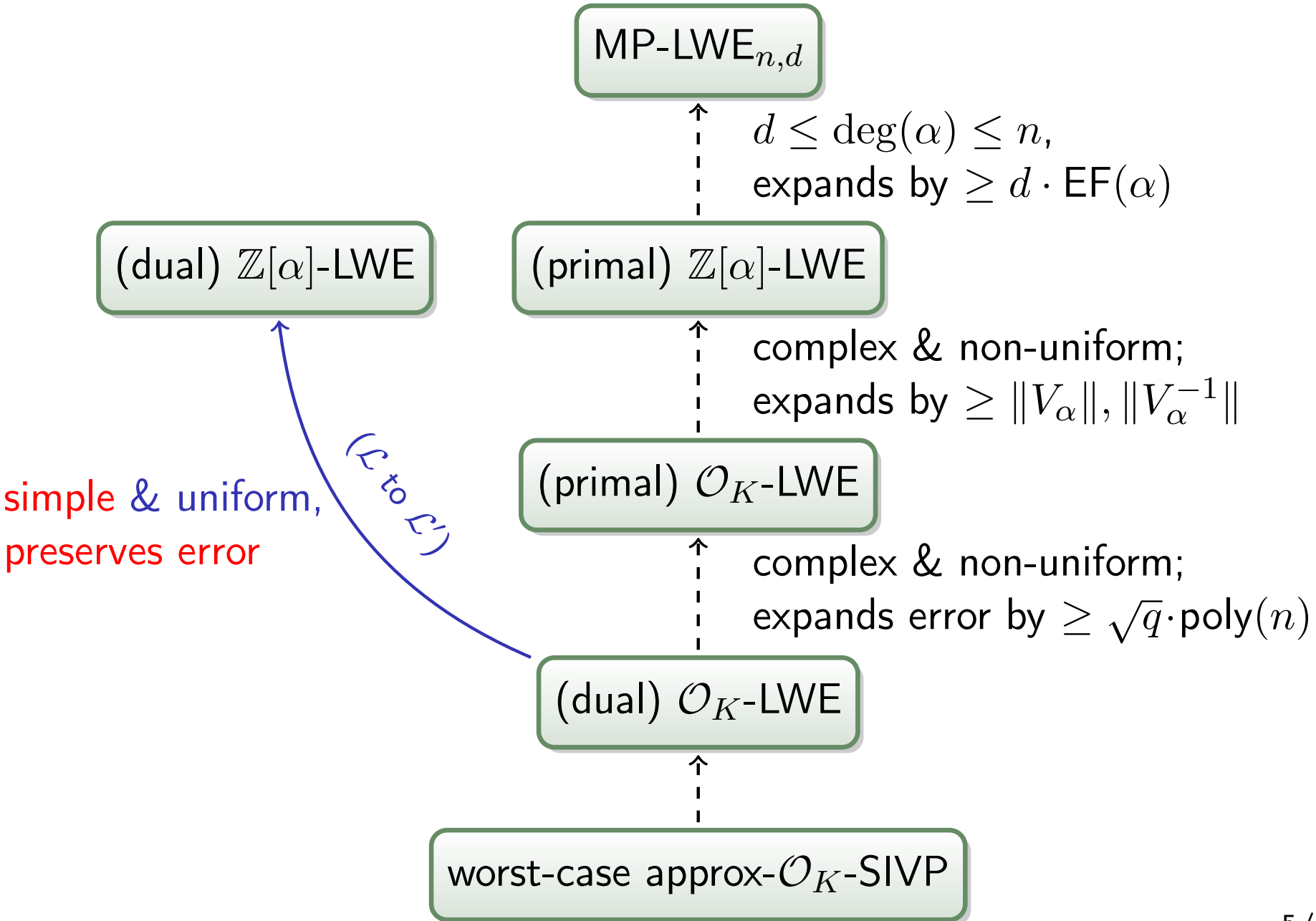

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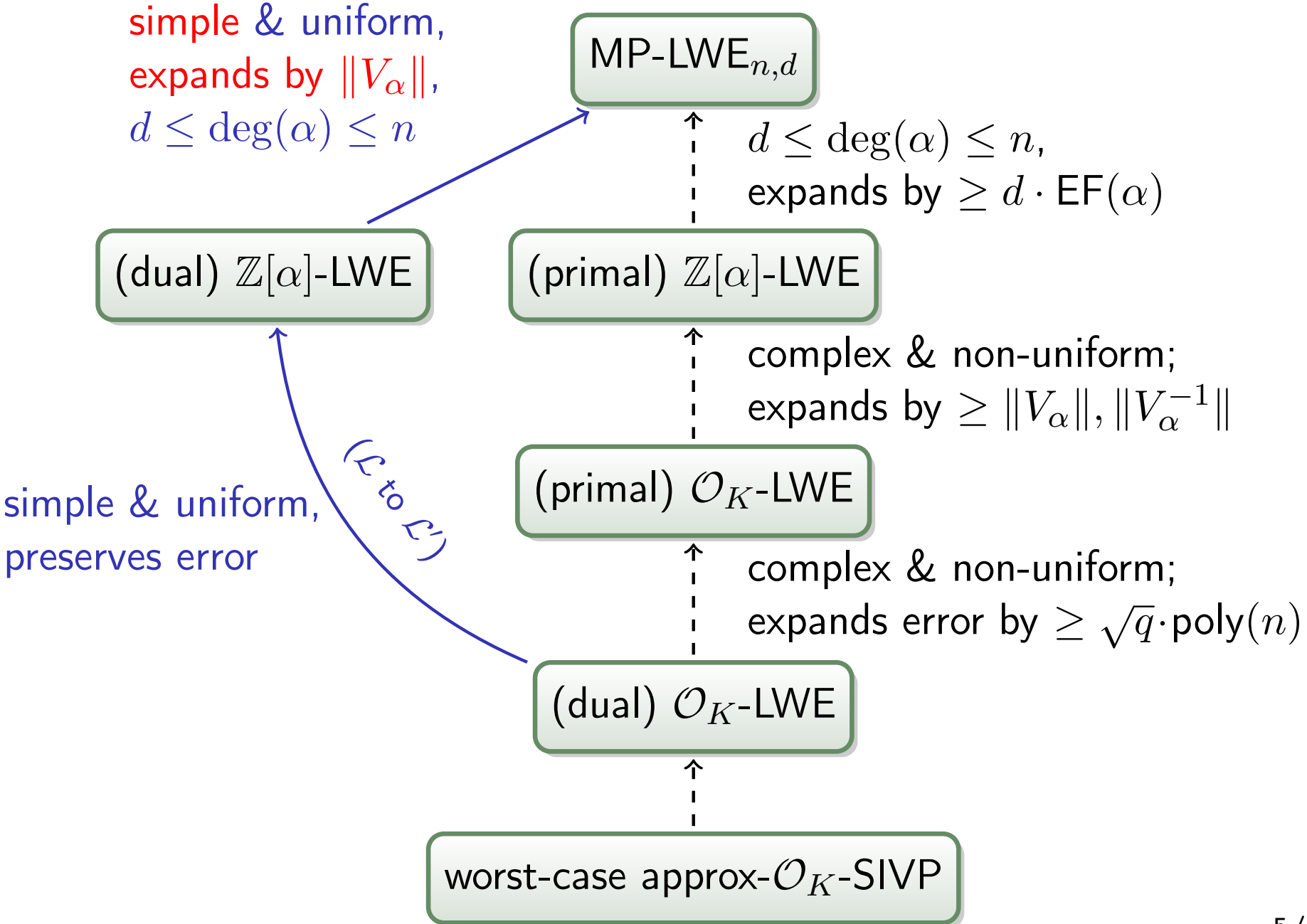

Ring-LWE and Variants

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Poly-LWE

- Same, but $R = \mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/f(x)$ and $s, a, s \cdot a \in R_q$ (no dual $R_q^\vee$).

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- Generalizes:

Ring-LWE	by taking $\mathcal{L} = \mathcal{O}_K$ to be the full ring of integers
Order-LWE	by taking $\mathcal{L} = \mathcal{O}$ to be an order of K
Poly-LWE	by taking $\mathcal{L} = \mathbb{Z}[\alpha]^\vee$ for some $\alpha \in \mathcal{O}_K$
Module-LWE	by allowing a, s to be vectors

Our $\mathcal{L}$-LWE Reductions

Theorem 1: $\mathcal{L}$ to $\mathcal{L}'$

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- Proof: $\mathcal{O}'$ is a rank-d $\mathcal{O}$-module. Keep just first coordinate of $b \approx s \cdot a$.

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MP-LWE

- For $s \in \mathbb{Z}_q^{<n+d-1}[x]$ and $a \in \mathbb{Z}_q^{<n}[x]$, the

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- Proof sketch: rest of the talk...

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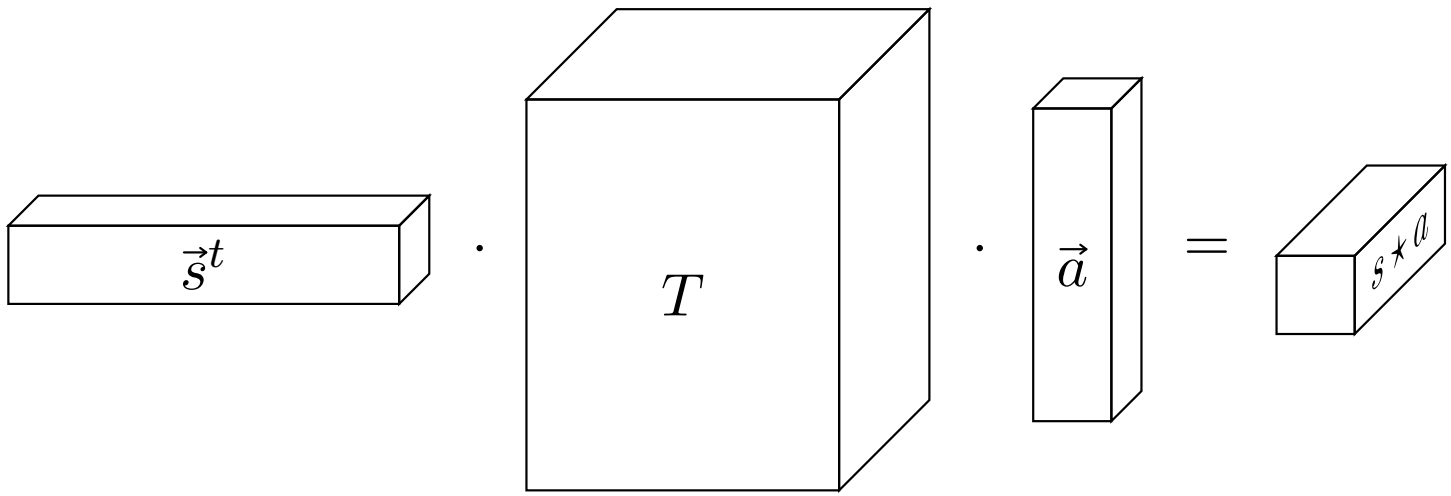

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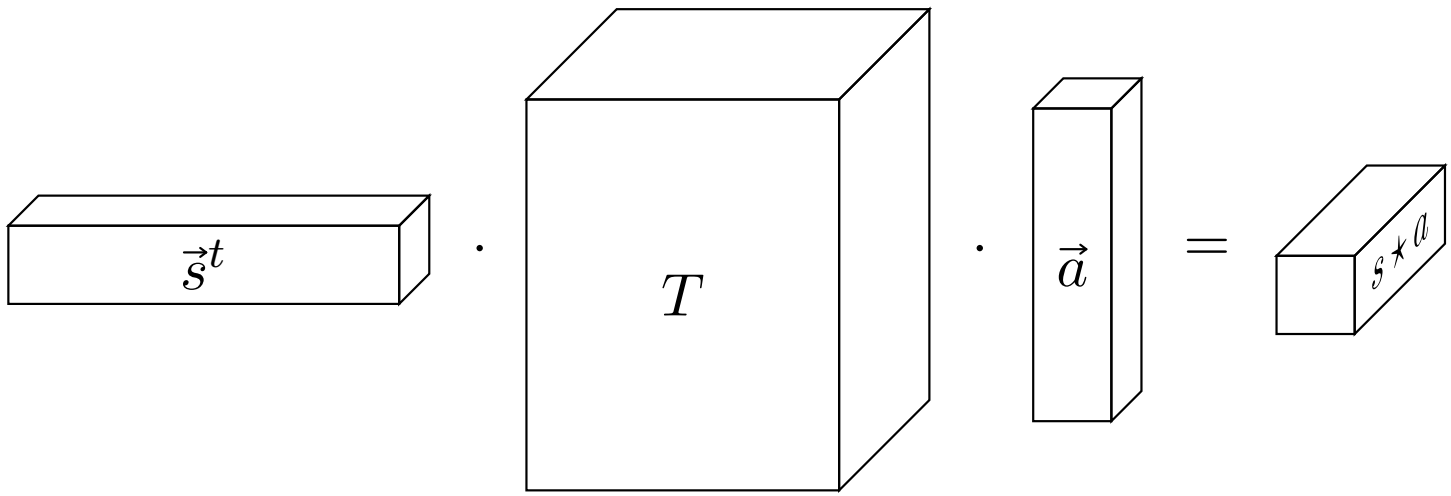
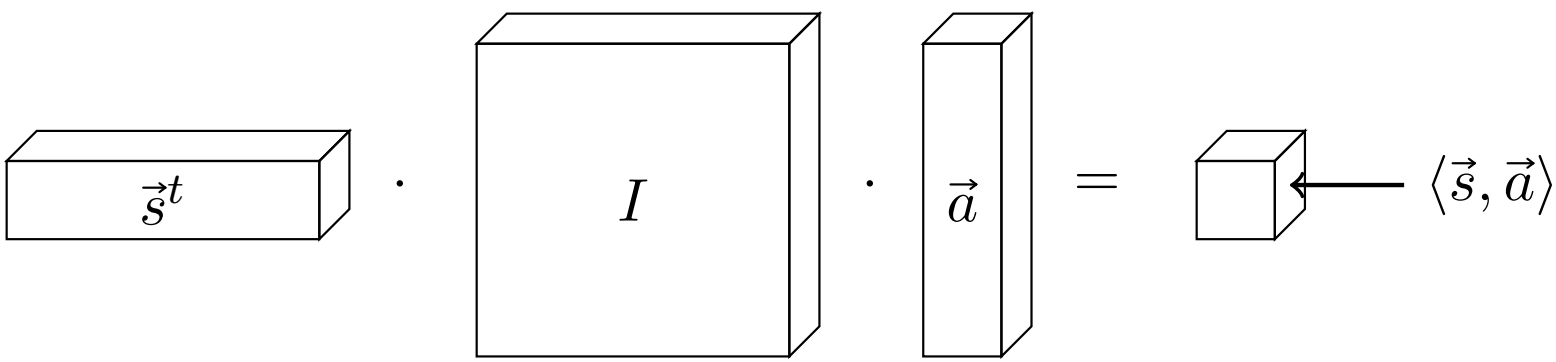
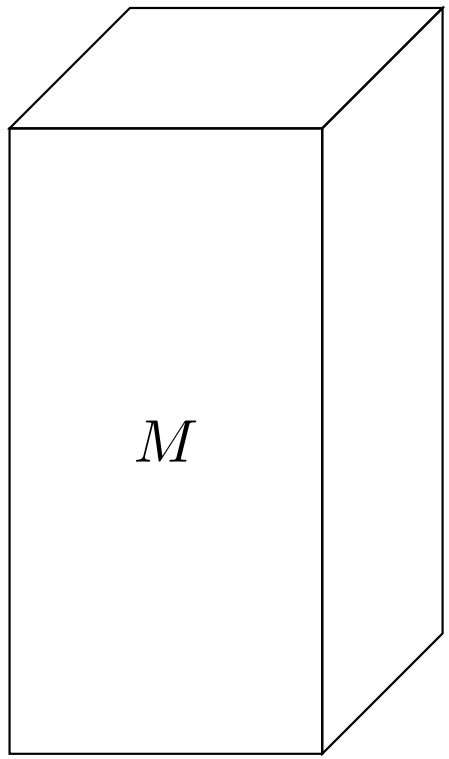


- E.g., plain LWE:



Middle-Product-LWE$_{n,d}$ Tensor



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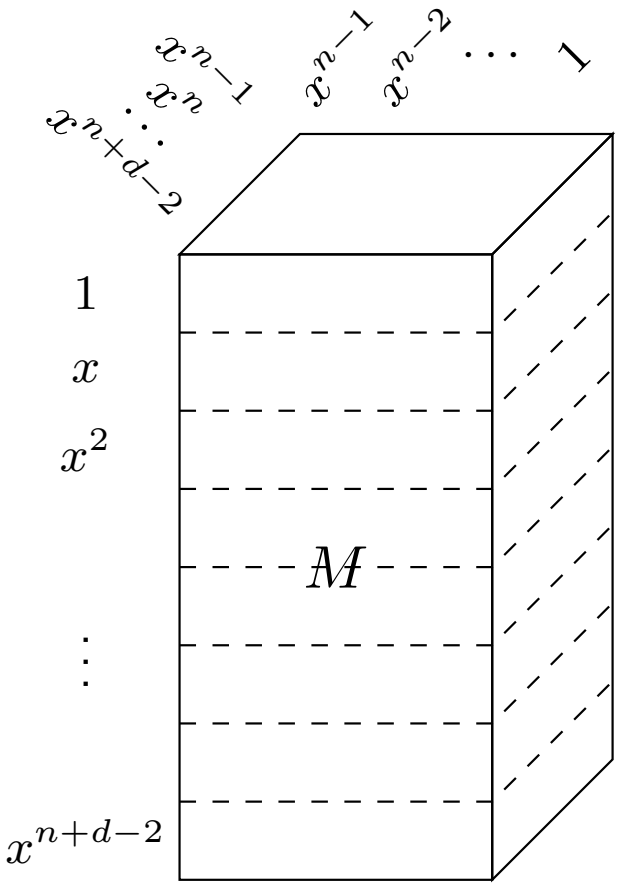

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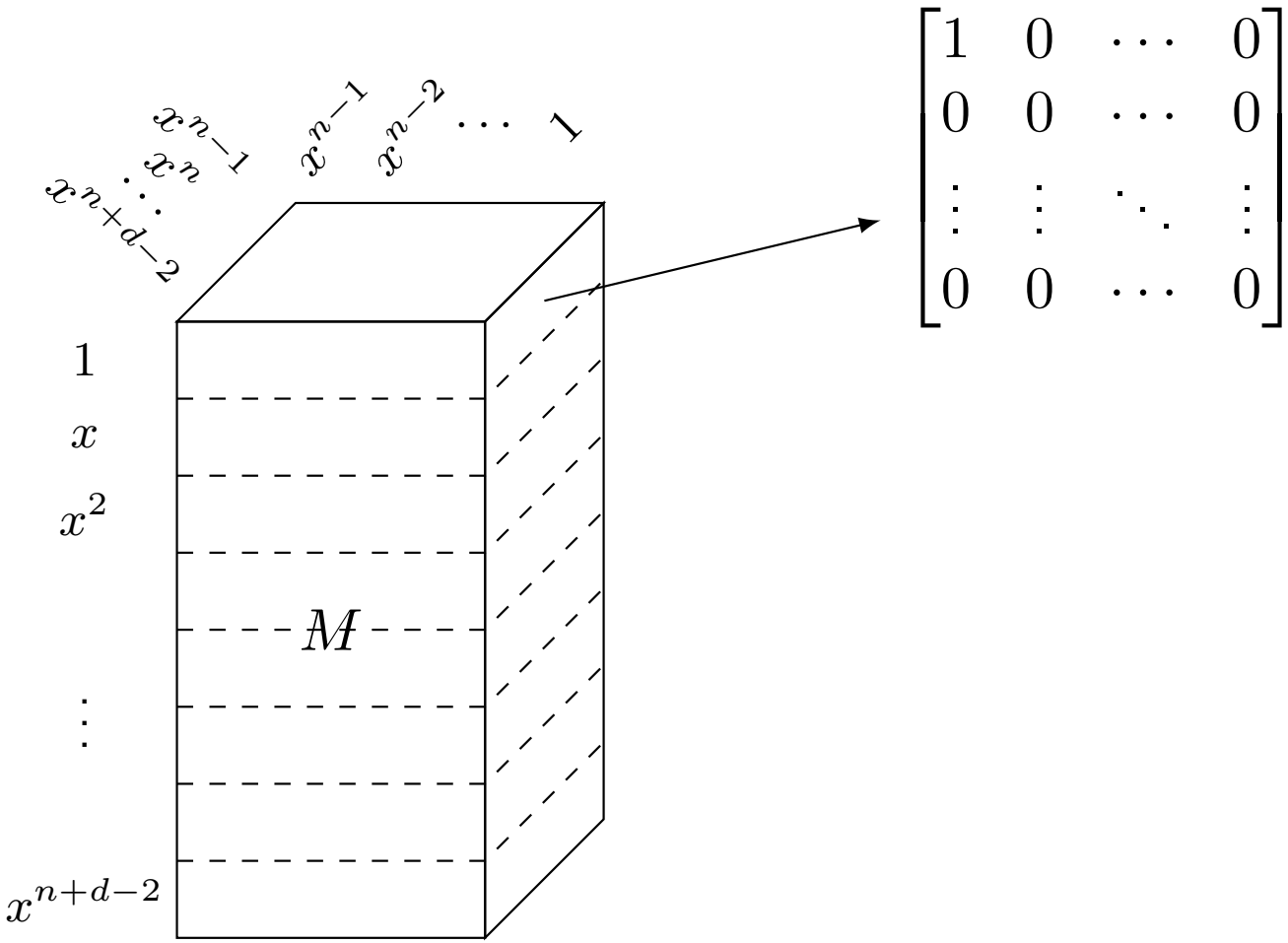

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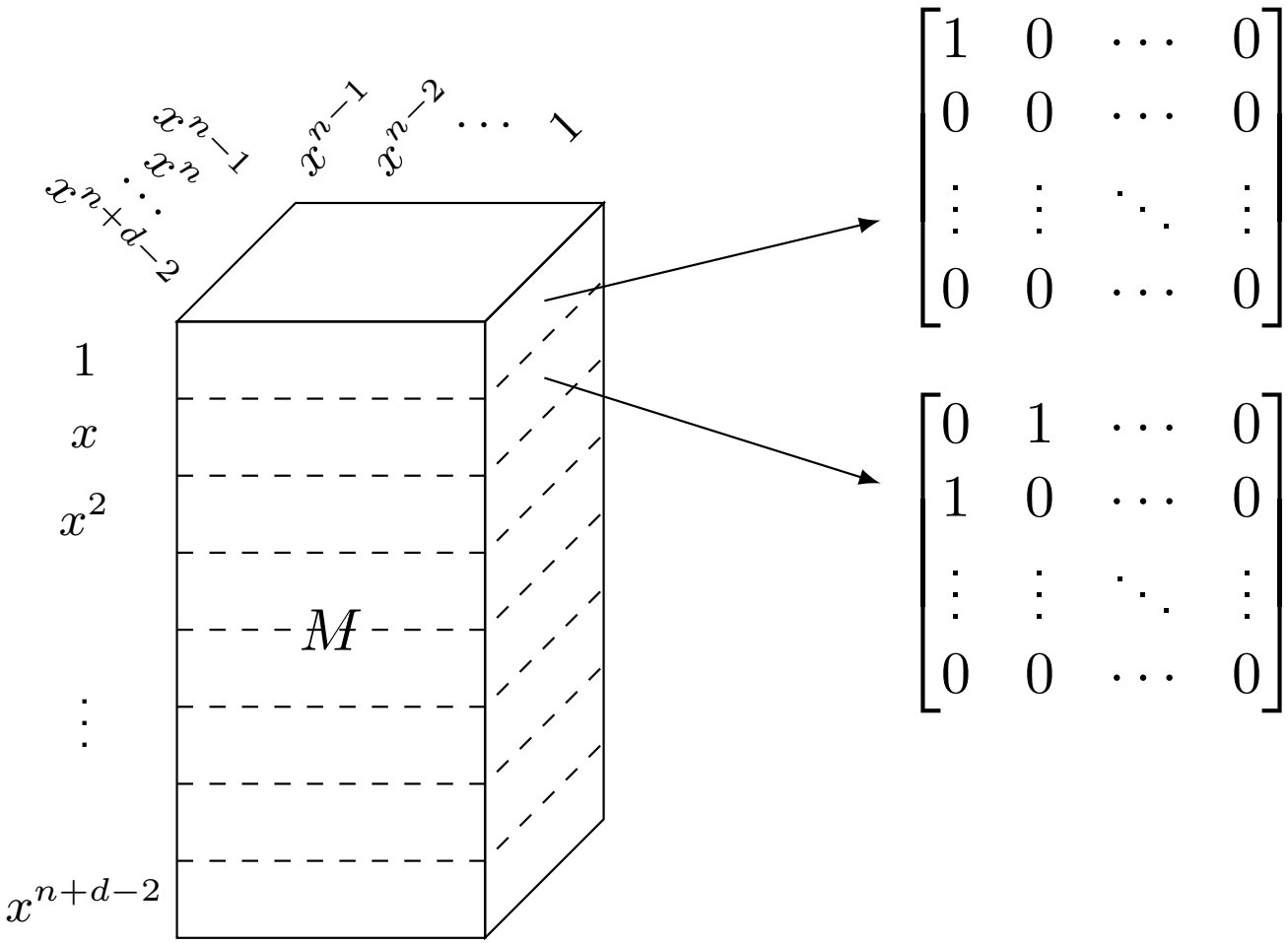

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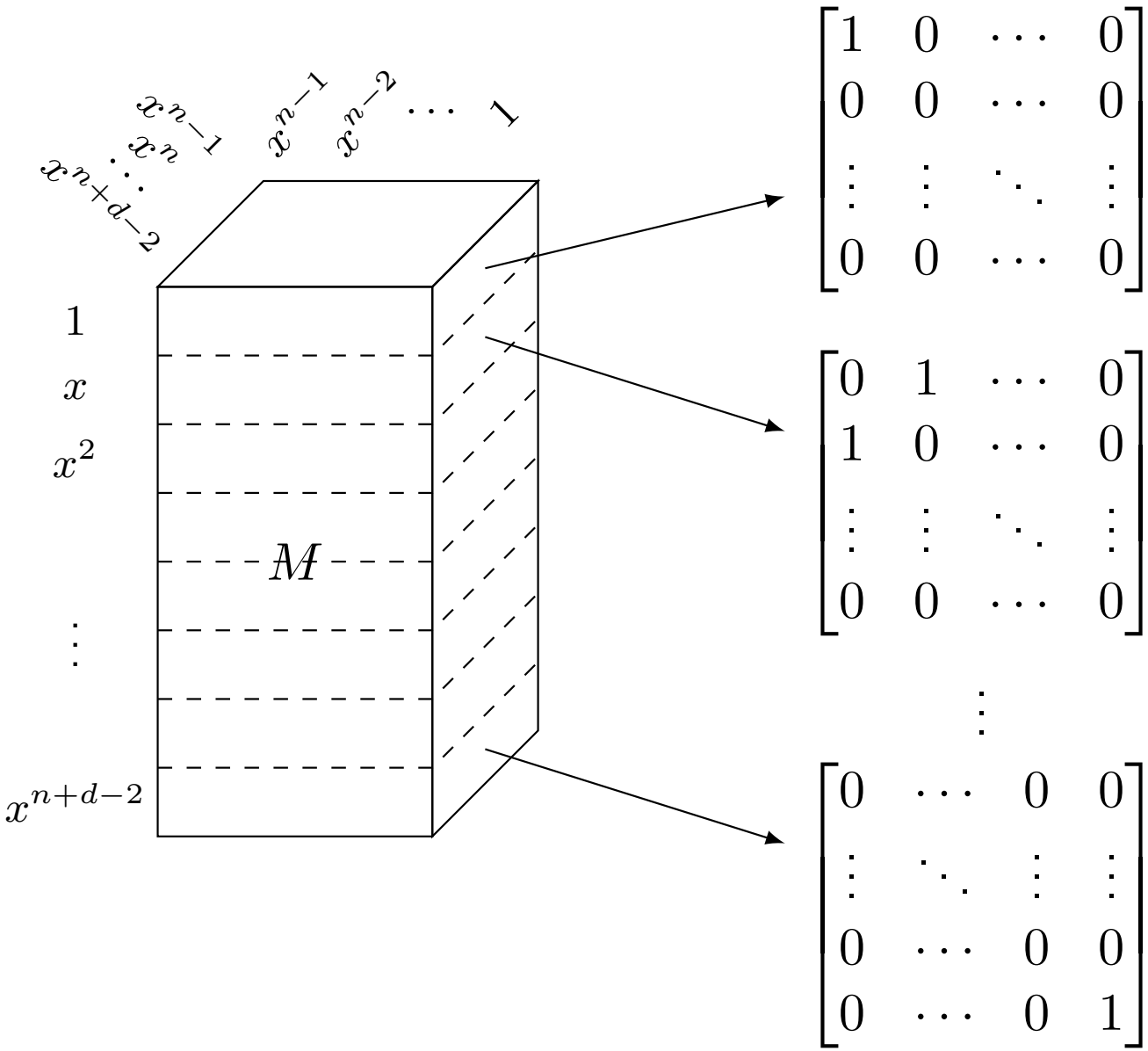

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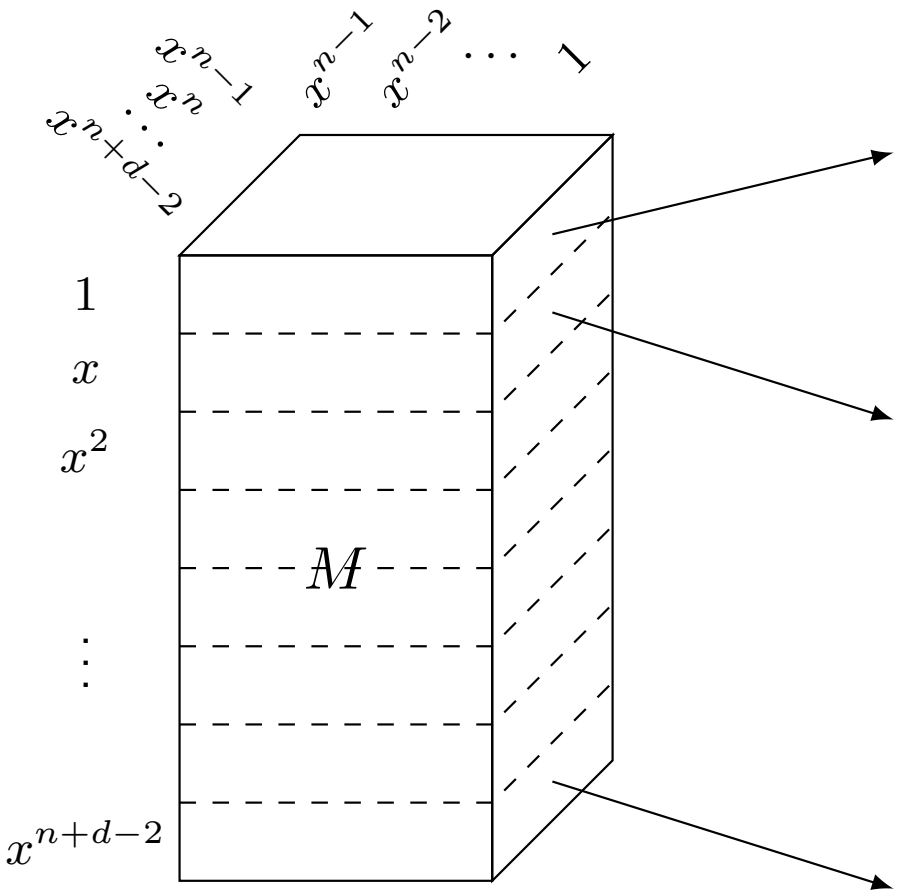

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

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$\vdots$

$$\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

a full basis of $d \times n$ Hankel matrices:

entry j, k given by $j + k$

(constant on anti-diagonals)

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$$T_{i,j,k} = \mathrm{Tr}(p_i^\vee \cdot p_j \cdot p_k) = \mathrm{Tr}(p_i^\vee \cdot \alpha^{j+k}),$$

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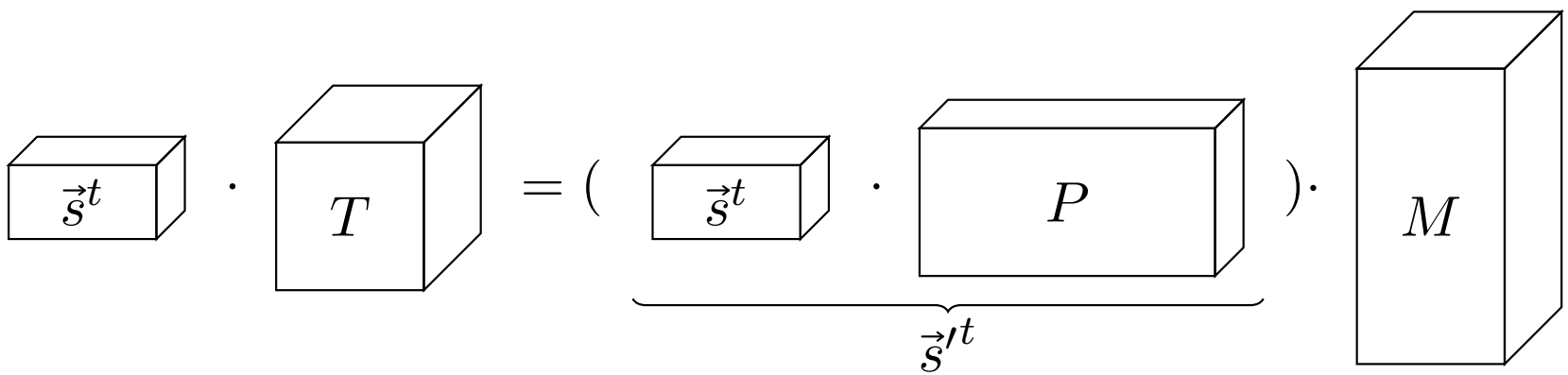

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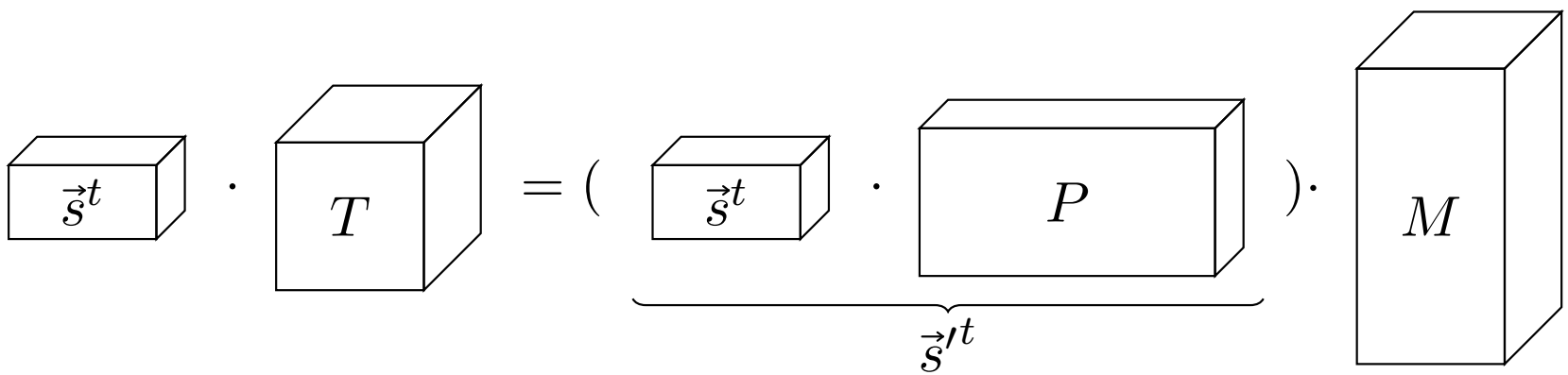

- Generally: T-LWE $\leq$ M-LWE for any T, M that factor as above.

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Thanks!

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