Complex and Quaternion-Valued Lattices for Digital Transmission

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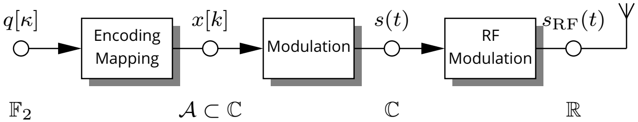
Work supported by Deutsche Forschungsgemeinschaft (DFG) under grant FI 982/12-1

Outline

1. **Complex-Valued Lattices in Communications**
   - System Model and Motivation
   - Introduction to Complex-Valued Lattices
   - Coded-Modulation Schemes based on Complex-Valued Lattices
   - Complex-Valued Lattice Reduction for Channel Equalization

2. **Quaternion-Valued Lattices in Communications**
   - System Model and Motivation
   - Introduction to Quaternion-Valued Lattices
   - Coded-Modulation Schemes based on Quaternion-Valued Lattices
   - Quaternion-Valued Lattice Reduction for Channel Equalization

Digital Transmission: System Model (I)

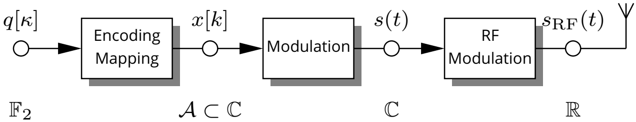


Digital Pulse-Amplitude Modulation: Transmitter

- transmission of sequence of bits $q[\kappa] \in \mathbb{F}_2$
- encoding and mapping
  - channel encoding over the finite field (Hamming space)
  - mapping to complex-valued signal constellation $\mathcal{A}$ (Euclidean space)
- modulation

$$s(t) = \sum_{k=-\infty}^{\infty} x[k]\, g(t - kT)$$

  - transition from discrete-time to continuous-time domain
  - transmit filter $g(t)$ (usually: band limitation)
  - symbol period T



Digital Pulse-Amplitude Modulation: Transmitter

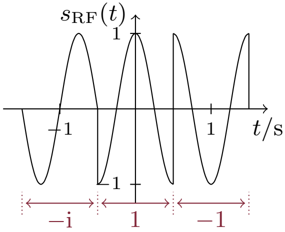
- radio-frequency (RF) modulation

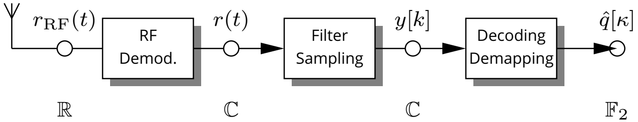
$$s_{\mathrm{RF}}(t) = |s(t)| \cdot \cos(2\pi f_{\mathrm{c}} t + \arg\{s(t)\})$$

  - complex signal modulated onto the real-valued carrier (frequency f_{c})
  - amplitude of RF signal given by $|s(t)|$
  - phase of RF signal given by $\arg\{s(t)\}$

Example

- $\mathcal{A} = \{1, -1, \mathrm{i}, -\mathrm{i}\}$
- $g(t) = \mathrm{rect}(t/1\,\mathrm{s})$
  $\Rightarrow |s(t)| = 1$
- $x[k] = [\ldots, -\mathrm{i}, 1, -1, \ldots]$
- $f_{\mathrm{c}} = 1\,\mathrm{Hz}$

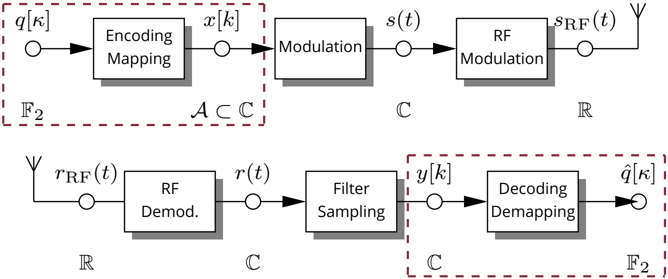




Digital Pulse-Amplitude Modulation: Receiver

- RF demodulation
  - equivalent complex baseband signal obtained from RF receive signal
- receive filter and sampling
  - usually *matched filter* $g^*(-t)$
    $\rightarrow$ maximization of signal-to-noise ratio (SNR)
  - sampling
    $\rightarrow$ transition from continuous-time to discrete-time domain
- decoding and demapping
  - channel decoding w.r.t. coded-modulation scheme
    $\rightarrow$ interaction between channel code and signal constellation
  - demapping to estimated source bits

Discrete-Time Equivalent-Complex-Baseband Domain



Discrete-Time Equivalent Complex Baseband (ECB)

- digital signal processing usually performed in
  - baseband domain (complex signals)
  - discrete-time domain
- discrete-time ECB transmission model with
  - transmit symbols $x[k]$
  - receive symbols $y[k]$
  - complex-valued channel model
    $\rightarrow$ equivalent representation of distortions in ECB domain

Additive White Gaussian Noise (AWGN) Channel

$$\underbrace{y[k]}_{\mathrm{Re}\{y[k]\}+\mathrm{Im}\{y[k]\}\,\mathrm{i}} = \underbrace{x[k]}_{\mathrm{Re}\{x[k]\}+\mathrm{Im}\{x[k]\}\,\mathrm{i}} + \underbrace{n[k]}_{\mathrm{Re}\{n[k]\}+\mathrm{Im}\{n[k]\}\,\mathrm{i}}$$

- discrete-time complex-valued noise $n[k]$
- noise samples usually zero-mean Gaussian with some variance σ_n^2
- samples are white over time (i.i.d.)
- transmission performance depends on SNR expressed as σ_x^2/σ_n^2

Block-Based Transmission over the AWGN Channel

$$\underline{\boldsymbol{y}} = \underline{\boldsymbol{x}} + \underline{\boldsymbol{n}}$$

- sequence of transmit/receive symbols and noise split into blocks

$$\underbrace{[y_1, y_2, \ldots, y_{N_\mathrm{b}}]}_{\underline{\boldsymbol{y}}} = \underbrace{[x_1, x_2, \ldots, x_{N_\mathrm{b}}]}_{\underline{\boldsymbol{x}}} + \underbrace{[n_1, n_2, \ldots, n_{N_\mathrm{b}}]}_{\underline{\boldsymbol{n}}}$$

- for brevity, block index omitted

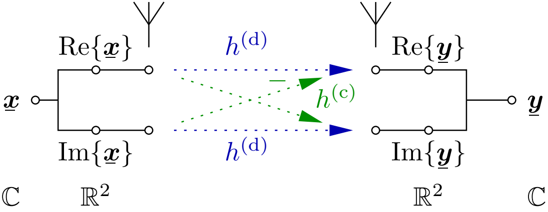
Single-Input/Single-Output (SISO) Block-Fading Channel

$$\underbrace{\underline{y}}_{\mathrm{Re}\{\underline{y}\}+\mathrm{Im}\{\underline{y}\}\,\mathrm{i}} = \underbrace{h}_{\mathrm{Re}\{h\}+\mathrm{Im}\{h\}\,\mathrm{i}} \cdot \underbrace{\underline{x}}_{\mathrm{Re}\{\underline{x}\}+\mathrm{Im}\{\underline{x}\}\,\mathrm{i}} + \underbrace{\underline{n}}_{\mathrm{Re}\{\underline{n}\}+\mathrm{Im}\{\underline{n}\}\,\mathrm{i}}$$

- equivalently described by real-valued matrix equation

$$\begin{bmatrix} \mathrm{Re}\{\underline{y}\} \\ \mathrm{Im}\{\underline{y}\} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathrm{Re}\{h\} & -\mathrm{Im}\{h\} \\ \mathrm{Im}\{h\} & \mathrm{Re}\{h\} \end{bmatrix}}_{\begin{bmatrix} h^{(\mathrm{d})} & -h^{(\mathrm{c})} \\ h^{(\mathrm{c})} & h^{(\mathrm{d})} \end{bmatrix}} \begin{bmatrix} \mathrm{Re}\{\underline{x}\} \\ \mathrm{Im}\{\underline{x}\} \end{bmatrix} + \begin{bmatrix} \mathrm{Re}\{\underline{n}\} \\ \mathrm{Im}\{\underline{n}\} \end{bmatrix}$$

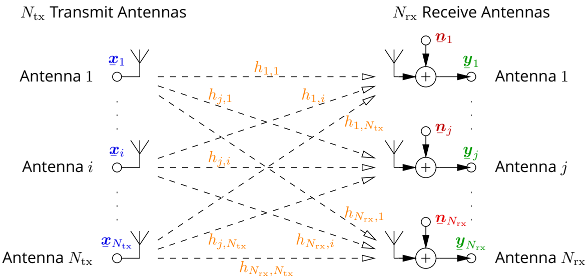
- complex-valued fading factor $h = h^{(\mathrm{d})} + h^{(\mathrm{c})}\,\mathrm{i}$
  $\rightarrow$ usually complex Gaussian



$h^{(\mathrm{d})}$ direct link

$h^{(\mathrm{c})}$ cross link

Multiple-Input/Multiple-Output (MIMO) Block-Fading Channel



$\rightarrow$ wireless multi-antenna transmission (same time and frequency)

Representation via MIMO System Equation:

$$\underbrace{\begin{bmatrix} \underline{\boldsymbol{y}}_1 \\ \vdots \\ \underline{\boldsymbol{y}}_{N_{\mathrm{rx}}} \end{bmatrix}}_{\substack{\boldsymbol{Y} \\ \text{receive symbols}}} = \underbrace{\begin{bmatrix} h_{1,1} & \cdots & h_{1,N_{\mathrm{tx}}} \\ \vdots & \ddots & \vdots \\ h_{N_{\mathrm{rx}},1} & \cdots & h_{N_{\mathrm{rx}},N_{\mathrm{tx}}} \end{bmatrix}}_{\substack{\boldsymbol{H} \\ \text{channel matrix}}} \cdot \underbrace{\begin{bmatrix} \underline{\boldsymbol{x}}_1 \\ \vdots \\ \underline{\boldsymbol{x}}_{N_{\mathrm{tx}}} \end{bmatrix}}_{\substack{\boldsymbol{X} \\ \text{transmit symbols}}} + \underbrace{\begin{bmatrix} \underline{\boldsymbol{n}}_1 \\ \vdots \\ \underline{\boldsymbol{n}}_{N_{\mathrm{rx}}} \end{bmatrix}}_{\substack{\boldsymbol{N} \\ \text{noise}}}$$

Fields of Application

- channel coding
- signal constellations

$\left.\right\}$ coded modulation

- lattice-reduction-aided MIMO equalization

$\Rightarrow$ design of coded-modulation schemes for AWGN or MIMO scenarios

Problem

- transmit and receive signals are complex-valued
- lattice theory is most often considered over real numbers

$\Rightarrow$ complex-valued lattices are required

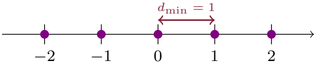
Definition of a Lattice [CS '99, Fis '02]

$$\boldsymbol{\Lambda}(\boldsymbol{G}) = \left\{ \sum_{v=1}^{V} \boldsymbol{g}_v \zeta_v \mid \zeta_v \in \mathbb{Z} \right\}$$

- created by generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_V] \in \mathbb{R}^{U \times V}$
- defined over integer ring $\mathbb{Z}$
- *infinite* set of points (vectors) over U-dimensional Euclidean space
- Abelian group w.r.t. addition

Integer ring $\mathbb{Z}$

- Euclidean ring
  - division with small remainder possible
  - Euclidean algorithm well-defined
- two nearest neighbors
- squared minimum distance $d_{\min}^2 = 1$



$\Rightarrow$ how can we extend the definition to complex lattices?

Generalized Definition

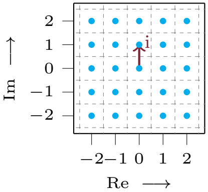
[Ste '19]

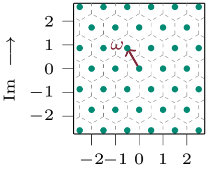
$$\mathbf{\Lambda}(\boldsymbol{G}) = \left\{ \sum_{v=1}^{V} \boldsymbol{g}_v \zeta_v \mid \zeta_v \in \mathbb{I} \right\}$$

- complex generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \dots, \boldsymbol{g}_V] \in \mathbb{C}^{U \times V}$
- defined over complex integer ring $\mathbb{I}$

Complex Integer Rings

- Gaussian integers $\mathbb{G} = \mathbb{Z} + \mathbb{Z}\,\mathrm{i}$
  - Euclidean ring
  - four nearest neighbors
  - squared minimum distance $d^2_{\min} = |i|^2 = 1$
  - isomorphic to 2D real-valued integer lattice $\mathbb{Z}^2$
- Eisenstein integers $\mathbb{E} = \mathbb{Z} + \mathbb{Z}\,\omega$
  - $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ *Eisenstein unit* (sixth root of unity)
  - Euclidean ring
  - six nearest neighbors
  - squared minimum distance $d^2_{\min} = |\omega|^2 = 1$
  - isomorphic to 2D hexagonal lattice $\boldsymbol{A}_2$





Voronoi Constellations

[For '89, Fis '02]

$$\mathcal{A} = \mathbf{\Lambda}_{\mathrm{a}} \cap \mathcal{R}_{\mathrm{V}}(\mathbf{\Lambda}_{\mathrm{b}})$$

- signal-point lattice $\mathbf{\Lambda}_{\mathrm{a}}$
- Voronoi region $\mathcal{R}_{\mathrm{V}}(\mathbf{\Lambda}_{\mathrm{b}})$ of boundary lattice $\mathbf{\Lambda}_{\mathrm{b}}$ (w.r.t. origin)

Complex-Valued Construction

[Ste '19]

- $\mathbf{\Lambda}_{\mathrm{a}}$ complex integer ring ($\mathbb{G}$ or $\mathbb{E}$)
- $\mathbf{\Lambda}_{\mathrm{b}} = \Theta\mathbf{\Lambda}_{\mathrm{a}}$, $\Theta \in \mathbf{\Lambda}_{\mathrm{a}}$, scaled version

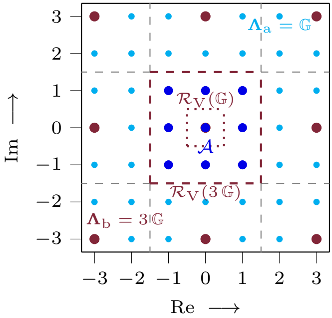
$\rightarrow$ constellation with $M = |\Theta|^2$ signal points

$\rightarrow$ *periodic extensions* with modulo function

$$\mathrm{mod}_{\mathbf{\Lambda}_{\mathrm{b}}}\{x\} = x - \mathcal{Q}_{\mathbf{\Lambda}_{\mathrm{b}}}\{x\} \in \mathcal{R}_{\mathrm{V}}(\mathbf{\Lambda}_{\mathrm{b}})$$

$\mathcal{Q}_{\mathbf{\Lambda}_{\mathrm{b}}}\{\cdot\}$ quantization w.r.t. Voronoi cells of $\mathbf{\Lambda}_{\mathrm{b}}$

$\Rightarrow$ how to combine with channel coding?



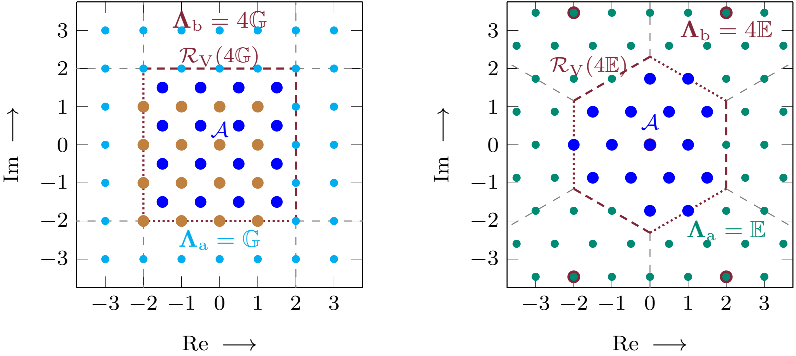
$|\Theta|^2 = 9$ signal points

Bit-Interleaved Coded Modulation (BICM) [CTB '98]

- close-to-optimum scheme for high SNRs
- channel coding over $\mathbb{F}_2$ (binary codes), encoded bits are interleaved
- blocks of b bits mapped to $M = 2^b$-ary constellation

$\Rightarrow$ Voronoi construction: $\mathbf{\Lambda}_\mathrm{a} = \mathbb{G}$ or $\mathbf{\Lambda}_\mathrm{a} = \mathbb{E}$, $\mathbf{\Lambda}_\mathrm{b} = \sqrt{M}\mathbf{\Lambda}_\mathrm{a}$

$\rightarrow$ edges have to be handled properly, offset for $\mathbf{\Lambda}_\mathrm{a} = \mathbb{G}$

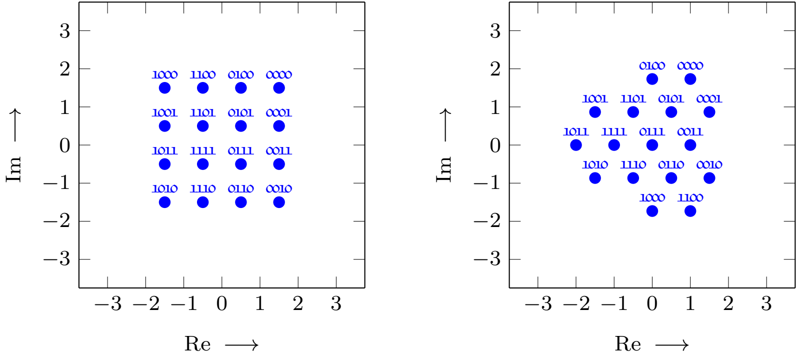


Bit-Interleaved Coded Modulation: Bit Labeling

- Gray labeling required for close-to-optimum performance
- neighbored signal points may only differ in one bit
- for square QAM ($\mathbf{\Lambda}_\mathrm{a} = \mathbb{G}$), Gray labeling well-known

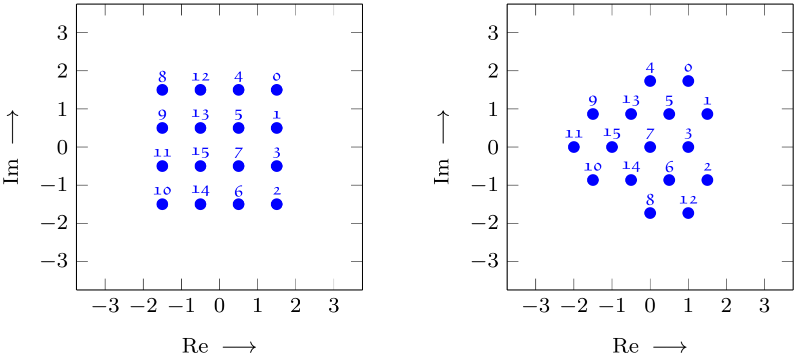
$\Rightarrow$ Gray labeling for Eisenstein-based constellations?

$\rightarrow$ unfortunately, only pseudo-Gray labeling (1.5 bits on average)



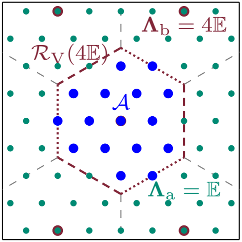
Alternative Coded-Modulation Strategies for Binary Labeling

- multilevel coding over Gaussian/Eisenstein integers [FHSG '18, SRFF '19]
- channel coding over $\mathbb{F}_{2^b}$
  - b bits are mapped to corresponding 2^b finite-field elements
  - finite-field elements are directly mapped to signal points
  - optimum performance possible, but usually high complexity



Gaussian (QAM) vs. Eisenstein Constellations

- hexagonal lattice: packing gain
- hexagonal boundaries: shaping gain



Asymptotic Gains

[CS '99, Fis '02]

- packing gain: $0.6247\ \mathrm{dB}$
- shaping gain: $0.1671\ \mathrm{dB}$
- total gain: $0.6247\ \mathrm{dB} + 0.1671\ \mathrm{dB} = 0.7918\ \mathrm{dB}$

Variance of the Constellation

M	$\sigma^2_{x,\mathbb{G}}$	$\sigma^2_{x,\mathbb{E}}$	$\sigma^2_{x,\mathbb{E}}/\sigma^2_{x,\mathbb{G}}$
4	0.5	0.75	1.5
16	2.5	2.25	0.9
64	10.5	9	0.8571
256	42.5	35.625	0.8382
1024	170.5	142.3125	0.8347
4096	682.5	568.9688	0.8337
∞			$0.8\bar{3} \mathrel{\widehat{=}} -0.7918\ \mathrm{dB}$

so far: $\Theta \in \mathbb{R}$, particularly $\Theta = \sqrt{2^b}$

now: consider $\mathbf{\Lambda}_\mathrm{b} = \Theta \mathbf{\Lambda}_\mathrm{a}$, $\Theta \in \mathbf{\Lambda}_\mathrm{a} \subset \mathbb{C}$:

- scaling of Voronoi region by $|\Theta|$
- rotation of Voronoi region by $\arg\{\Theta\}$

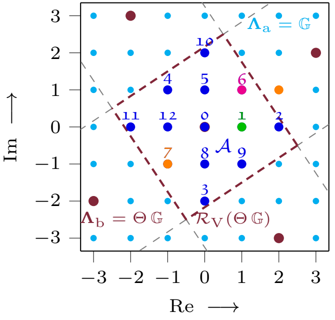
$\rightarrow$ how to choose Θ conveniently?

Gaussian Primes [Hub '94b, CS '99]

Gaussian integer $\Theta \in \mathbb{G}$ with

- $|\Theta|^2 = p$, p prime
- $\mathrm{rem}_4\{p\} = 1$ $\rightarrow$ e.g., $p = 5, 13, 17, \ldots$

$\Theta = 3 + 2\,\mathrm{j}$



$|\Theta|^2 = 3^2 + 2^2 = 13$

$\mathcal{A} \simeq \mathbb{F}_p = \{0, 1, \ldots, 11, 12\}$

Example: $6 + 1 = 7$

Gaussian Prime Constellations

- constellation with $|\Theta|^2 = p$ signal points
- isomorphism between finite field $\mathbb{F}_p$ and constellation $\mathcal{A}$

$\Rightarrow$ *straightforward coded modulation* (code over $\mathbb{F}_p \Leftrightarrow p$-ary constellation)

Eisenstein Primes [Hub '94a, CS '99]

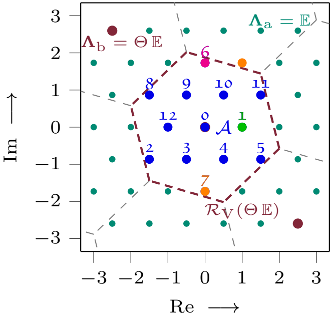
Eisenstein integer $\Theta \in \mathbb{E}$ with

- $|\Theta|^2 = p$, p prime
- $\mathrm{rem}_6\{p\} = 1 \quad \rightarrow$ e.g., $p = 7, 13, 19, \ldots$

$\rightarrow$ hexagonal Voronoi region

- scaled by $|\Theta|$
- rotated by $\arg\{\Theta\}$

$\Theta = 4 + 1\omega$



$|\Theta|^2 = 4^2 - \sqrt{3}^2 = 13$

$\mathcal{A} \simeq \mathbb{F}_p = \{0, 1, \ldots, 11, 12\}$

Example: $6 + 1 = 7$

Eisenstein Prime Constellations

- constellation with $|\Theta|^2 = p$ signal points
- isomorphism between finite field $\mathbb{F}_p$ and constellation $\mathcal{A}$

$\Rightarrow$ packing and shaping gain over Gaussian prime constellations

Algebraic Lattice Constellations (III)

Advantages

- straightforward coded modulation
- joint arithmetic over Hamming and Euclidean space
- suited in combination with MIMO equalization schemes

  $\rightarrow$ linear combinations of signal points

  $\updownarrow$

  linear combinations of codewords

Disadvantages

- no direct mapping from bits to constellation points
- conversion from $\mathbb{F}_2$ to $\mathbb{F}_p$ required (*modulus conversion* [FU '98, Fis '02])

  $\rightarrow$ mapping from blocks of μ bits to ν p-ary symbols
- accompanied by conversion/rate loss ($2^\mu < p^\nu$)
- error propagation if decoding fails

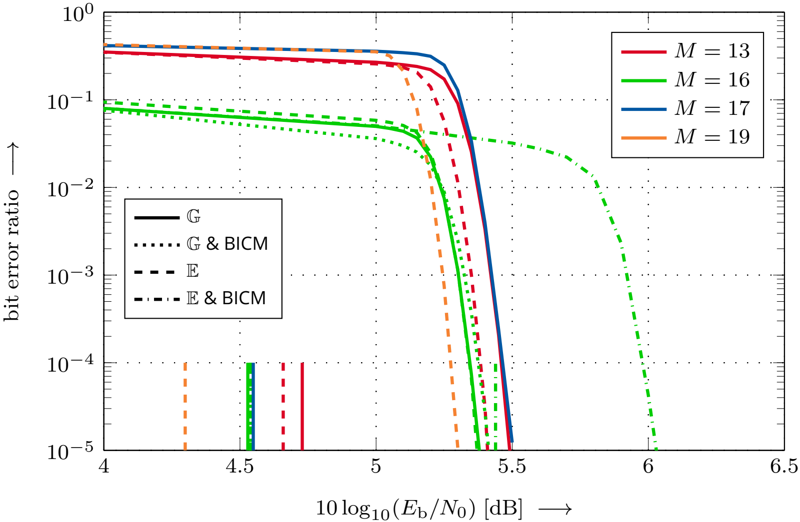
Transmission Scenario

- AWGN channel
  - transmission of binary source symbols (bits)
  - 3 information bits per symbol (code rates adjusted accordingly)
  - block-based transmission (64800 bits per block)
  - averaged over 10^5 codewords and related noise samples
- 16-ary Gaussian/Eisenstein constellation
  - combination with BICM (binary code)
  - combination with 2^b-ary channel code
- algebraic signal constellations
  - 13-ary Gaussian and Eisenstein prime constellation
  - 17-ary Gaussian prime constellation
  - 19-ary Eisenstein prime constellation

  $\rightarrow$ conversion to/from elements of $\mathbb{F}_p$

Channel Coding

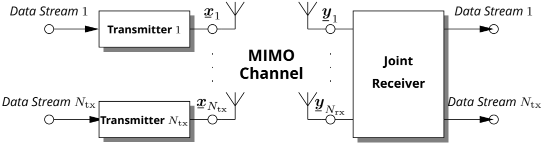
- binary/non-binary low-density parity-check (LDPC) codes over $\mathbb{F}_q$
  $\rightarrow$ subclass of irregular repeat-accumulate codes [KP '02]
- binary/non-binary belief-propagation decoding over field $\mathbb{F}_q$ [CML+ '12]



E_b: energy per information bit
N_0: noise power spectral density

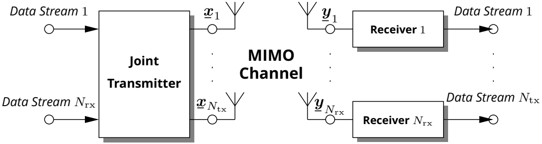
vertical lines: related capacities

MIMO Multiple-Access Channel (Multiuser Uplink)

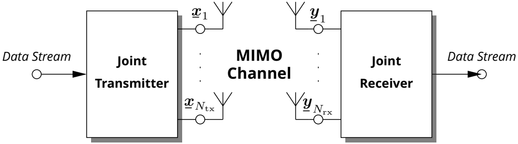


here:
multiuser uplink
considered

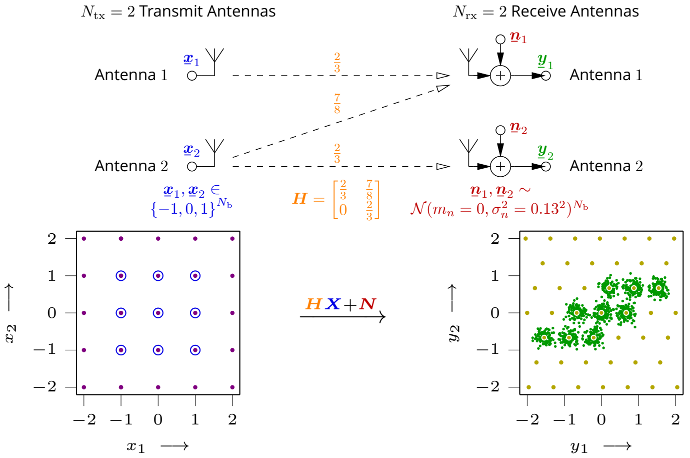
MIMO Broadcast Channel (Multiuser Downlink)



Point-to-Point MIMO Transmission (Single User)



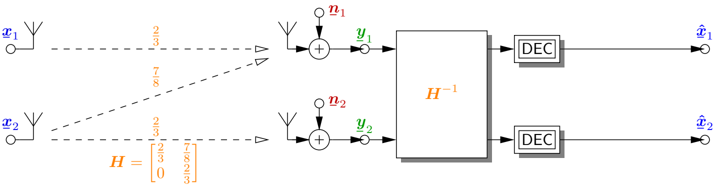
Example: 2×2 MIMO Transmission (Real-Valued)

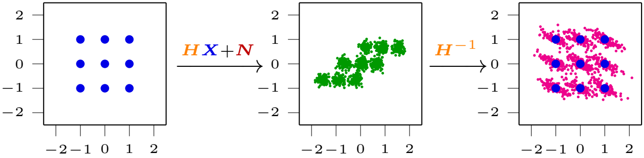


transmit symbols drawn from regular arrangement of points (*subset of integer ring*)

$\Downarrow$

receive symbols drawn from regular arrangement of points (plus noise)

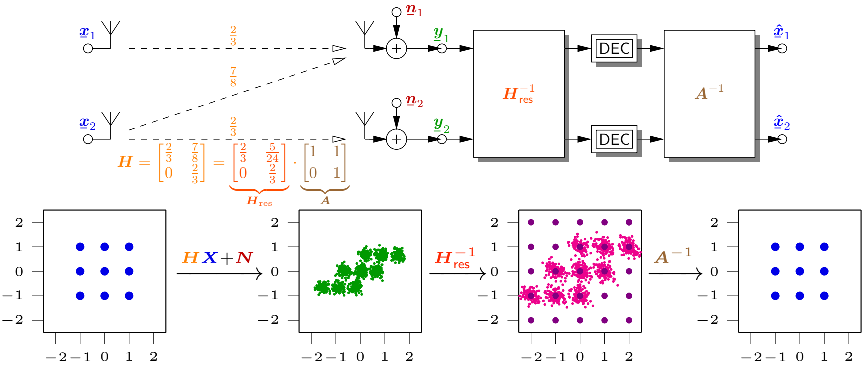




Linear Equalization by Channel Inversion

- receive symbols multiplied by H^{-1} (in general: pseudoinverse)
- works over real or complex numbers
- problem: *large noise enhancement*

  $\rightarrow$ decoding will often fail



LRA Equalization for Real-Valued Channels

[YW '02, WF '03]

- MIMO channel matrix factorized into $H = \underbrace{H_{\text{res}}}_{\text{drawn from } \cancel{\mathbb{R}}\ \mathbb{C}} \cdot \underbrace{A}_{\text{drawn from } \cancel{\mathbb{Z}}\ \mathbb{G}, \mathbb{E}}$
- linear equalization of non-integer part
- decoding w.r.t. signal-point lattice $\Lambda_a = \cancel{\mathbb{Z}}\ \mathbb{G}, \mathbb{E}$
- equalization of integer part (*no noise enhancement!*)

$\Rightarrow$ *generalization to complex-valued lattices* [Ste '19]

Classical Zero-Forcing (ZF) Factorization Approach

[YW '02, WF '03]

$$\boldsymbol{H} = \underbrace{\boldsymbol{H}_{\mathrm{res}}}_{\mathbb{C}^{N_{\mathrm{rx}} \times N_{\mathrm{tx}}}} \cdot \underbrace{\boldsymbol{A}}_{\substack{\boldsymbol{\Lambda}_{\mathrm{a}}^{N_{\mathrm{tx}} \times N_{\mathrm{tx}}} \\ |\det(\boldsymbol{A})|=1}}$$

- integer matrix $\boldsymbol{A}$ unimodular
- channel matrix $\boldsymbol{H}$ considered as generator matrix of lattice $\boldsymbol{\Lambda}(\boldsymbol{H})$
- $\boldsymbol{A}$ describes change to *more suited* lattice basis, i.e., $\boldsymbol{\Lambda}(\boldsymbol{H}) = \boldsymbol{\Lambda}(\boldsymbol{H}_{\mathrm{res}})$

$\Rightarrow$ algorithms for lattice basis reduction are suited

Minimum Mean-Square Error (MMSE) Extension

[WBKK '04]

factorization of *augmented channel matrix* $\boldsymbol{\mathcal{H}}$ according to

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \boldsymbol{H} \\ \sqrt{\zeta}\boldsymbol{I} \end{bmatrix} = \underbrace{\boldsymbol{\mathcal{H}}_{\mathrm{res}}}_{\mathbb{C}^{(N_{\mathrm{rx}}+N_{\mathrm{tx}}) \times N_{\mathrm{tx}}}} \cdot \underbrace{\boldsymbol{A}}_{\substack{\boldsymbol{\Lambda}_{\mathrm{a}}^{N_{\mathrm{tx}} \times N_{\mathrm{tx}}} \\ |\det(\boldsymbol{A})|=1}}, \qquad \zeta = \frac{\sigma_n^2}{\sigma_x^2}$$

$\Rightarrow$ equalization matrix for MMSE linear equalization of the non-integer part

Lenstra-Lenstra-Lovász (LLL) Reduction

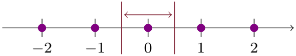
[LLL '82]

- polynomial-time algorithm
- can be seen as some kind of Euclidean algorithm for matrices
- operates on QR decomposition $\boldsymbol{G} = \boldsymbol{Q}\boldsymbol{R} \in \mathbb{R}^{U \times V}$
  - $\boldsymbol{Q}$ matrix with orthogonal columns
  - $\boldsymbol{R}$ upper triangular matrix with unit main diagonal

Reduction Criteria

generator matrix $\boldsymbol{G} = \boldsymbol{Q}\boldsymbol{R}$ is LLL-reduced, if

- size reduction condition



$$|r_{l,v}| < \frac{1}{2}\,, \qquad 1 \leq l < v \leq V$$

- Lovász condition (quality parameter $\delta \in (1/4, 1]$)

$$\|\boldsymbol{q}_v\|^2 \geq (\delta - |r_{v-1,v}|^2)\|\boldsymbol{q}_{v-1}\|^2\,, \qquad v = 2, \ldots, V$$

fulfilled

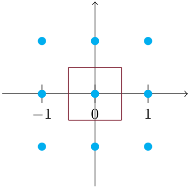
Complex Lenstra-Lenstra-Lovász (CLLL) Reduction

[GLM '09]

- Lovász condition unchanged
- size reduction condition adapted to

$$|\mathrm{Re}\{r_{l,v}\}| \leq \frac{1}{2} \ \cap \ |\mathrm{Im}\{r_{l,v}\}| \leq \frac{1}{2}$$

- quality parameter $\delta \in (1/2, 1]$



Generalization of Size-Reduction Criterion

- size reduction condition

$$\mathcal{Q}_{\mathbb{I}}\{r_{l,v}\} = 0\,, \qquad 1 \leq l < v \leq V$$

- minimum quality parameter δ is maximum squared quantization error

$\Rightarrow$ Eisenstein-LLL with condition $\mathcal{Q}_{\mathbb{E}}\{r_{l,v}\} = 0$ and $\delta \in (1/3, 1]$ [SF '15]

LLL Reduction: Performance

- LLL reduction operates on QR decomposition of (reduced) generator matrix
- actually required: basis vectors of reduced lattice basis as short as possible

Minkowski Reduction

[Min '91]

- successive determination of shortest lattice vectors that form a basis of the lattice
- NP-hard *shortest vector problem* has to be solved in each step

$\Rightarrow$ optimum but high-complexity lattice-basis-reduction approach

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Minkowski Reduction: Criterion/Algorithm

[ZQW '12]

- consider lattice $\boldsymbol{\Lambda}(\boldsymbol{G})$ with generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_V] \in \mathbb{R}^{U \times V}$
- in step $v = 1, \ldots, V$, the basis vector $\boldsymbol{g}_v = \boldsymbol{G}\boldsymbol{\zeta}_v$ is chosen such that

$$\boldsymbol{\zeta}_v = \underset{\boldsymbol{\zeta}_v \in \mathbb{Z}^V}{\operatorname{argmin}} \|\boldsymbol{G} \cdot [\zeta_1, \ldots, \zeta_V]^{\mathsf{T}}\|$$

- the related integer vector $\boldsymbol{\zeta}_v$ additionally has to fulfill

$$\gcd\{\zeta_v, \ldots, \zeta_V\} = 1$$

$\rightarrow$ unimodular integer matrix describes related change of basis

Minkowski Reduction: Generalization

- greatest common divisor calculated by Euclidean algorithm
- Euclidean algorithm defined over all Euclidean rings

$\Rightarrow$ generalization to lattices over $\mathbb{G}$ and $\mathbb{E}$ possible [Ste '19]

$\Rightarrow$ restriction to unimodular integer matrices really necessary?

Full-Rank Relaxation [ZNEG '14, FCS '16]

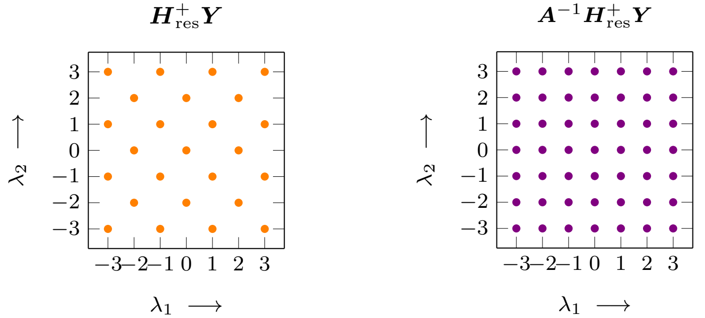
$$\boldsymbol{H} = \underbrace{\boldsymbol{H}_{\mathrm{res}}}_{\mathbb{C}^{N_{\mathrm{rx}} \times N_{\mathrm{tx}}}} \cdot \underbrace{\boldsymbol{A}}_{\substack{\boldsymbol{\Lambda}_{\mathrm{a}}^{N_{\mathrm{tx}} \times N_{\mathrm{tx}}} \\ \mathrm{rank}(\boldsymbol{A}) = N_{\mathrm{tx}}}}$$

Related Factorization Problem: Successive Minima Problem

- consider lattice $\boldsymbol{\Lambda}(\boldsymbol{G})$ with generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_V] \in \mathbb{R}^{U \times V}$
- find V shortest linearly independent lattice vectors (*successive minima*)
- related integer vectors $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_V$ form integer matrix $\boldsymbol{A}$
- NP-hard problem but efficient algorithms exist [DKWZ '15, FCS '16]

$\Rightarrow$ straightforward adaption to lattices over $\mathbb{G}$ and $\mathbb{E}$ [Ste '19]

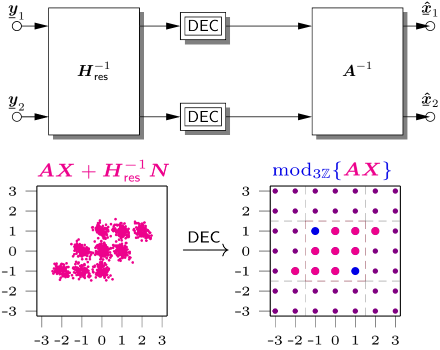
Unimodularity Constraint (II)



Real-Valued 2×2 Example (Noise Neglected)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad |\det(\boldsymbol{A})| = 2$$

- after non-integer equalization via $\boldsymbol{H}_{\mathrm{res}}^{+}$: sublattice of $\mathbb{Z}^2$
  $\rightarrow$ detection/decoding still possible
- after integer equalization via $\boldsymbol{A}^{-1}$: original lattice $\mathbb{Z}^2$ restored



Decoding of Linear Combinations

- at decoder inputs: cascade

$$\boldsymbol{H}_{\text{res}}^{-1}(\boldsymbol{HX}+\boldsymbol{N}) = \boldsymbol{AX} + \boldsymbol{H}_{\text{res}}^{-1}\boldsymbol{N}$$

$\rightarrow$ integer linear combinations $\boldsymbol{AX} \in \boldsymbol{\Lambda}_{\text{a}}^{N_{\text{tx}} \times N_{\text{b}}}$

- linear codes: linear combinations of codewords form codewords
- problem: coding performed over $\mathbb{F}_p$

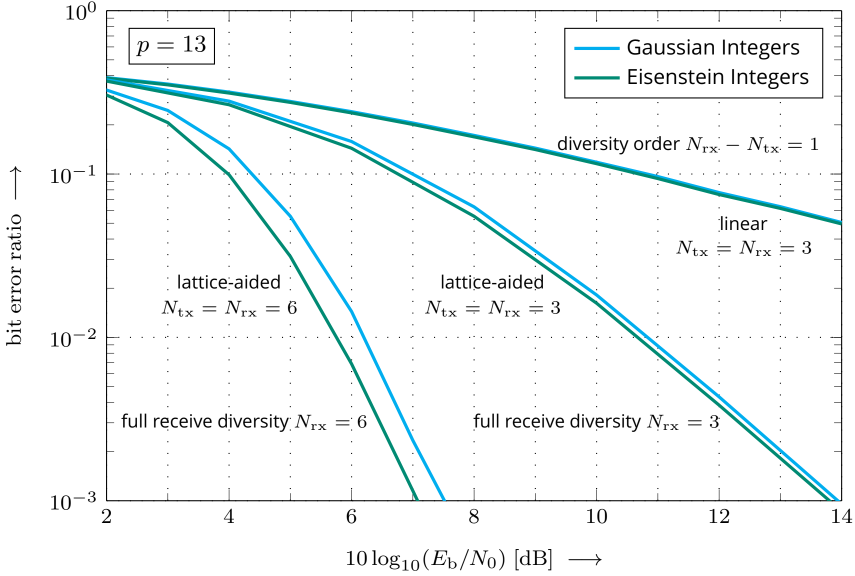
$\Rightarrow$ *joint arithmetic required!* [ZNEG '14]

Related Coded-Modulation Strategies

- based on algebraic constellations (real/complex-valued)
- utilize equivalence with signal points from $\mathcal{A}$ in modulo arithmetic
- two different strategies to handle integer interference:
  - integer forcing: resolved over finite-field in modulo arithmetic [ZNEG '14]
  - lattice-aided receiver: recovers the original linear combinations [Ste '19]
    $\Rightarrow$ *coded variant of the LRA equalization philosophy*

Transmission Scenario

- MIMO multiple-access channel
  - *Rayleigh fading* on each link (i.i.d. complex Gaussian)
  - channel matrix constant over codeword duration (*block fading*)
  - results averaged over many channel realizations (and users)
- lattice-aided receiver
  - 13-ary Gaussian and Eisenstein prime constellations
  - optimum channel factorization calculated (successive minima)
  - MMSE criterion
- coded modulation: properties from AWGN scenario
  - non-binary low-density parity-check (LDPC) codes over $\mathbb{F}_p$
  - 3 information bits per symbol


$p = 13$
Gaussian Integers
Eisenstein Integers
diversity order $N_{\mathrm{rx}} - N_{\mathrm{tx}} = 1$
linear
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 3$
lattice-aided
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 6$
lattice-aided
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 3$
full receive diversity $N_{\mathrm{rx}} = 6$
full receive diversity $N_{\mathrm{rx}} = 3$
bit error ratio $\longrightarrow$
10^0
10^{-1}
10^{-2}
10^{-3}
2
4
6
8
10
12
14
$10\log_{10}(E_{\mathrm{b}}/N_0)$ [dB] $\longrightarrow$

Part 1: Summary

- introduction to complex-valued transmission
- complex-valued lattices and related Voronoi constellations
- coded-modulation schemes
  - bit-interleaved coded modulation (BICM)
  - 2^b-ary coded modulation
  - coded modulation over algebraic signal constellations
- lattice-reduction-aided equalization for MIMO transmission
  - factorization approach (lattice basis reduction/unimodular integer matrix)
  - lattice-basis-reduction criteria (LLL/Minkowski)
  - relaxation of unimodularity constraint/successive minima problem
  - coded modulation using algebraic signal constellations
  - factorization gain of the Eisenstein integers

Complex Numbers $\mathbb{C}$

$$c = \underbrace{\mathrm{Re}\{c\}}_{\in\mathbb{R}} + \underbrace{\mathrm{Im}\{c\}}_{\in\mathbb{R}}\,\mathrm{i}$$

- field extension of $\mathbb{R}$
- imaginary unit $\mathrm{i} = \sqrt{-1}$

Quaternions $\mathbb{H}$

$$\begin{aligned} q &= \underbrace{(\mathrm{Re}\{q^{\{1\}})\} + \mathrm{Im}\{q^{\{1\}}\}\,\mathrm{i})}_{q^{\{1\}}\in\mathbb{C}} \quad + \quad \underbrace{(\mathrm{Re}\{q^{\{2\}}\} + \mathrm{Im}\{q^{\{2\}}\}\,\mathrm{i})}_{q^{\{2\}}\in\mathbb{C}}\;\mathrm{j} \\ &= \quad \underbrace{q^{(1)}}_{\in\mathbb{R}} + \underbrace{q^{(2)}}_{\in\mathbb{R}}\,\mathrm{i} \quad + \quad \underbrace{q^{(3)}}_{\in\mathbb{R}}\,\mathrm{j} + \underbrace{q^{(4)}}_{\in\mathbb{R}}\,\mathrm{k} \end{aligned}$$

- extension of $\mathbb{C}$
- imaginary units i, j, and $\mathrm{k} = \mathrm{i}\cdot\mathrm{j}$
- multiplication is non-commutative (*skew field*)

	i	j	k
i	-1	k	$-\mathrm{j}$
j	$-\mathrm{k}$	-1	i
k	j	$-\mathrm{i}$	-1

Quaternion-Valued Channel Models (I)

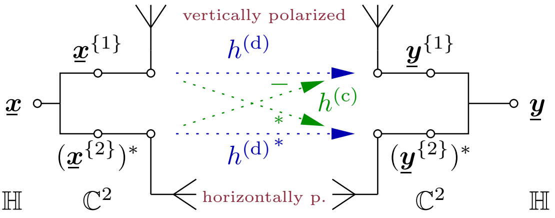
Quaternion-Valued SISO Block Fading (AWGN channel: $h = 1$)

$$\underbrace{\underline{\boldsymbol{y}}}_{\underline{\boldsymbol{y}}^{\{1\}}+\underline{\boldsymbol{y}}^{\{2\}}\,\mathrm{j}} = \underbrace{h}_{h^{\{1\}}+h^{\{2\}}\,\mathrm{j}} \cdot \underbrace{\underline{\boldsymbol{x}}}_{\underline{\boldsymbol{x}}^{\{1\}}+\underline{\boldsymbol{x}}^{\{2\}}\,\mathrm{j}} + \underbrace{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{n}}^{\{1\}}+\underline{\boldsymbol{n}}^{\{2\}}\,\mathrm{j}}$$

- equivalently described by complex-valued matrix equation

$$\begin{bmatrix} \underline{\boldsymbol{y}}^{\{1\}} \\ (\underline{\boldsymbol{y}}^{\{2\}})^* \end{bmatrix} = \underbrace{\begin{bmatrix} h^{\{1\}} & -h^{\{2\}} \\ (h^{\{2\}})^* & (h^{\{1\}})^* \end{bmatrix}}_{\begin{bmatrix} h^{(\mathrm{d})} & -h^{(\mathrm{c})} \\ (h^{(\mathrm{c})})^* & (h^{(\mathrm{d})})^* \end{bmatrix}} \begin{bmatrix} \underline{\boldsymbol{x}}^{\{1\}} \\ (\underline{\boldsymbol{x}}^{\{2\}})^* \end{bmatrix} + \begin{bmatrix} \underline{\boldsymbol{n}}^{\{1\}} \\ (\underline{\boldsymbol{n}}^{\{2\}})^* \end{bmatrix}$$

- realized via dual-polarized transmission [CLF ’14, LBX$^+$ ’16]



$h^{(\mathrm{d})}$ direct link

$h^{(\mathrm{c})}$ cross link

Quaternion-Valued System Equation

[IS '95, WWS '06]

$$\underline{\boldsymbol{y}} = h\underline{\boldsymbol{x}} + \underline{\boldsymbol{n}}$$

- transmit symbols $\underline{\boldsymbol{x}} = \underbrace{(\underline{\boldsymbol{x}}^{(1)} + \underline{\boldsymbol{x}}^{(2)}\,\mathrm{i})}_{\text{vertical}} + \underbrace{(\underline{\boldsymbol{x}}^{(3)} - \underline{\boldsymbol{x}}^{(4)}\,\mathrm{i})^*}_{\text{horizontal}}\,\mathrm{j}$

  $\rightarrow$ drawn from 4D signal constellation

- fading factor $h = \underbrace{(h^{(1)} + h^{(2)}\,\mathrm{i})}_{\text{direct gain}} + \underbrace{(h^{(3)} - h^{(4)}\,\mathrm{i})^*}_{\text{cross-polar gain}}\,\mathrm{j}$

  $\rightarrow$ four i.i.d. real-valued Gaussian coefficients

- additive noise $\underline{\boldsymbol{n}} = \underbrace{(\underline{\boldsymbol{n}}^{(1)} + \underline{\boldsymbol{n}}^{(2)}\,\mathrm{i})}_{\text{vertical noise}} + \underbrace{(\underline{\boldsymbol{n}}^{(3)} - \underline{\boldsymbol{n}}^{(4)}\,\mathrm{i})^*}_{\text{horizontal noise}}\,\mathrm{j}$

  $\rightarrow$ four i.i.d. real-valued Gaussian coefficients (AWGN)

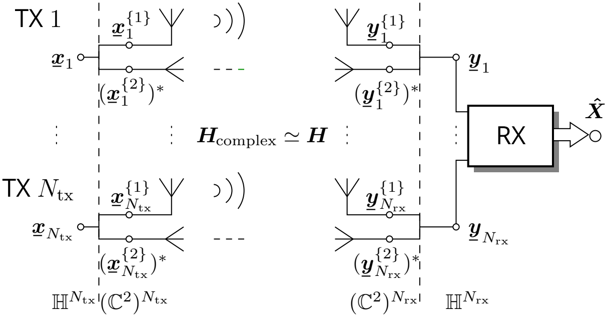
- receive symbols $\underline{\boldsymbol{y}} = \underbrace{(\underline{\boldsymbol{y}}^{(1)} + \underline{\boldsymbol{y}}^{(2)}\,\mathrm{i})}_{\text{vertical}} + \underbrace{(\underline{\boldsymbol{y}}^{(3)} - \underline{\boldsymbol{y}}^{(4)}\,\mathrm{i})^*}_{\text{horizontal}}\,\mathrm{j}$

Quaternion-Valued MIMO Channel

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{N}\,, \quad \text{with}$$

$$\boldsymbol{H} = \left[h_{l,k}\right]_{\substack{l=1,\dots,N_{\mathrm{rx}}\\ k=1,\dots,N_{\mathrm{tx}}}} \in \mathbb{H}^{N_{\mathrm{rx}}\times N_{\mathrm{tx}}}$$

N_{tx}-User MIMO Multiple-Access Channel with N_{rx} *Antenna Pairs* [SF '18]



Lipschitz Integers $\mathcal{L}$

- quaternions with components

$$l = \underbrace{l^{(1)}}_{\in\mathbb{Z}} + \underbrace{l^{(2)}}_{\in\mathbb{Z}}\,\mathrm{i} + \underbrace{l^{(3)}}_{\in\mathbb{Z}}\,\mathrm{j} + \underbrace{l^{(4)}}_{\in\mathbb{Z}}\,\mathrm{k}$$

- isomorphic to 4D integer lattice $\mathbb{Z}^4$

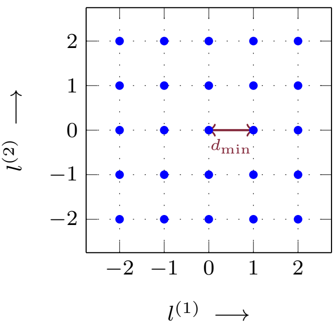
Important Properties [CS '99, CS '03]

- eight nearest neighbors
- squared minimum distance

$$d_{\min}^2 = 1^2 + 0^2 + 0^2 + 0^2 = 1$$

- $\mathcal{L}$ forms a *non-Euclidean* ring
  - no division with small remainder
  - Euclidean algorithm not defined

2D projection



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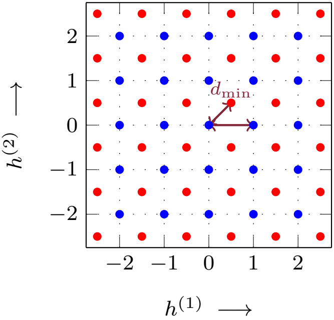
Hurwitz Integers $\mathcal{H}$

- quaternions with components

$$h = h^{(1)} + h^{(2)}\,\mathrm{i} + h^{(3)}\,\mathrm{j} + h^{(4)}\,\mathrm{k}\,,$$
$$(h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)}) \in \mathbb{Z}^4 \cup (\mathbb{Z} + 1/2)^4$$

- two subsets $\mathcal{H}_1 = \mathcal{L}$ and $\mathcal{H}_2 = \mathcal{L} + (1 + \mathrm{i} + \mathrm{j} + \mathrm{k})/2$
- isomorphic to 4D checkerboard lattice $\boldsymbol{D}_4$

2D projection



Important Properties [CS ’99, CS ’03]

- 24 nearest neighbors
- squared minimum distance

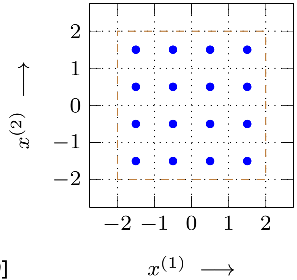
$$d^2_{\min} = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 = 1$$

- $\mathcal{H}$ forms a *Euclidean* ring
  - division with small remainder
  - Euclidean algorithm well-defined

Based on Lipschitz Integers $\mathcal{L}$

- QAM in each polarization
- Voronoi constellation over $\mathbb{G}$ (with offset)
- cardinalities $M = 4^2 = 16, 16^2 = 256, \dots$
- 4D Gray labeling possible
- separable into four 1D constellations

2D projection

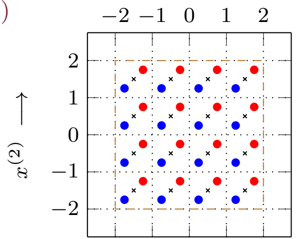


Based on Hurwitz Integers $\mathcal{H}$

[SF '18, SFFF '19]

- subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ with offset $\pm\frac{1}{4}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$
- enable one additional bit
  $\rightarrow M = 2 \cdot 16 = 32, 2 \cdot 256 = 512, \dots$
- same boundaries and minimum distance
- 4D Gray labeling not possible

$\rightarrow$ no straightforward application of bit-interleaved coded modulation (BICM)



Voronoi Constellations over Hurwitz Integers

$$\mathcal{A} = \mathcal{H} \cap \mathcal{R}_{\mathrm{V}}(\Theta\mathcal{H})$$

- can in theory be constructed
- problem: Voronoi region is 24-cell
  - 24 vertices
  - 96 edges
  - 96 faces

  $\rightarrow$ boundaries have to be handled properly

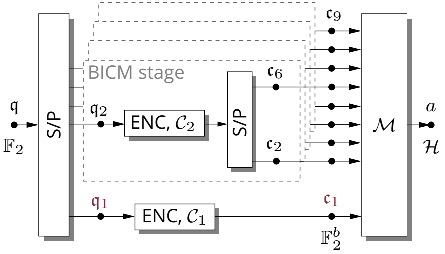
$\Rightarrow$ proposed Hurwitz constellations only benefit from packing gain

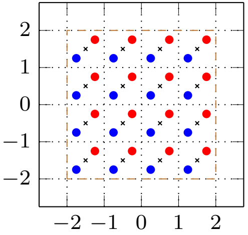
Algebraic Constellations over Lipschitz/Hurwitz Integers

- constellations based on Lipschitz primes
- constellations based on Hurwitz primes

$\Rightarrow$ part of current research

Two-Stage Transmitter for Hurwitz Constellations ($M = 512$) [SFFF '19]

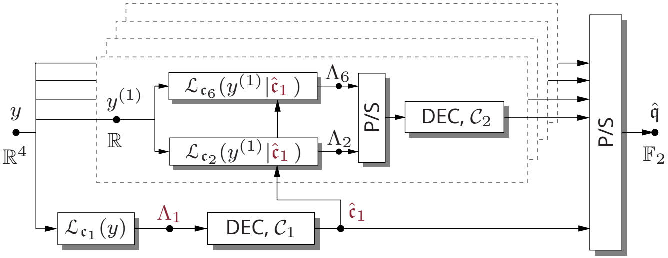




1. 4D stage: bit stream $\mathfrak{q}_1$ protected by low-rate code $\mathcal{C}_1$
   $\rightarrow$ encoded *offset bits* $\mathfrak{c}_1$ select via the mapping $\mathcal{M}$ between
   - Lipschitz subset $\mathcal{L}_1$ shifted by $-\frac{1}{4}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$
   - Lipschitz subset $\mathcal{L}_2$ shifted by $+\frac{1}{4}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$

2. 1D stage: conventional coded modulation in the subset
   - 4D signal treated per component (here: 4ASK)
   - BICM with mid-to-high-rate code $\mathcal{C}_2$ using Gray labeling

Two-Stage Decoder for Hurwitz Constellations $(M = 512)$ [SFFF '19]



1. 4D stage: decoding of offset bits

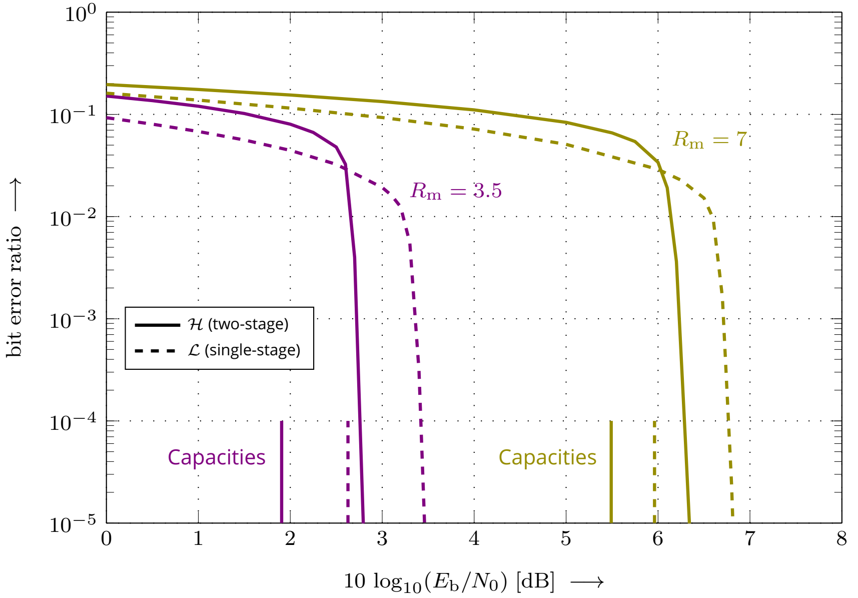
2. 1D stage: conventional decoding in the subset
   - already decoded offset bits $\hat{\mathfrak{c}}_1$ select between $\mathcal{L}_1$ and $\mathcal{L}_2$
   - individual metric calculation w.r.t. real-valued components $y^{(\cdot)}$

Numerical Simulations

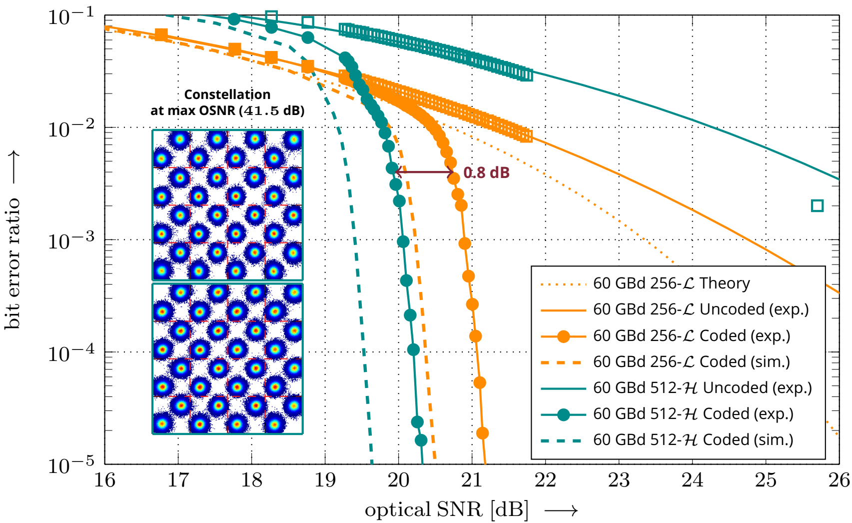
- numerical simulations using LDPC codes
  $\rightarrow$ binary irregular repeat-accumulate codes
- code length 64800
- scenarios with fixed modulation rate R_{m} (number of information bits)
- code rates according to capacity rule of multilevel coding [WFH '99]

Scenario	Approach	M	$R_{\mathrm{c},1}$	$R_{\mathrm{c},2}$
$R_{\mathrm{m}} = 3.5$	$\mathcal{H}$ (two-stage)	32	0.4909	0.7523
	$\mathcal{L}$ (single-stage)	16	–	0.8750
$R_{\mathrm{m}} = 7$	$\mathcal{H}$ (two-stage)	512	0.3103	0.8362
	$\mathcal{L}$ (single-stage)	256	–	0.8750

Hurwitz Constellations: Numerical Simulations (II)



Results of Fiber-Optic Transmission presented at ECOC 2019 [FSE+ '19]

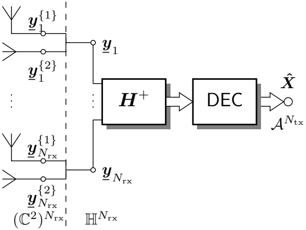


MIMO Multiple-Access Channel

- *quaternion-valued* linear equalization
- quaternion-valued channel inverse (ZF)

$$\boldsymbol{H}^{+} \in \mathbb{H}^{N_{\mathrm{tx}} \times N_{\mathrm{rx}}}$$

- MMSE criterion via augmented representation



Diversity Order (i.i.d. Gaussian Channel)

- $h_{l,k} = \mathrm{Re}\{h_{l,k}\} + \mathrm{Im}\{h_{l,k}\}\,\mathrm{i} \in \mathbb{C}$

  $\rightarrow$ two independent gains; i.e., diversity

  $$\Delta_{\mathrm{c,LIN}} = N_{\mathrm{rx}} - N_{\mathrm{tx}} + 1 \qquad (\text{i.e., } \Delta_{\mathrm{c,LIN}} = 1 \text{ for } N_{\mathrm{tx}} = N_{\mathrm{rx}})$$

- $h_{l,k} = h_{l,k}^{(1)} + h_{l,k}^{(2)}\,\mathrm{i} + h_{l,k}^{(3)}\,\mathrm{j} + h_{l,k}^{(4)}\,\mathrm{k} \in \mathbb{H}$

  $\rightarrow$ four independent gains; i.e., diversity

  $$\Delta_{\mathrm{q,LIN}} = 2\,(N_{\mathrm{rx}} - N_{\mathrm{tx}} + 1) \qquad (\text{i.e., } \Delta_{\mathrm{q,LIN}} = 2 \text{ for } N_{\mathrm{tx}} = N_{\mathrm{rx}})$$

Quaternion-Valued LRA MIMO Equalization (I)

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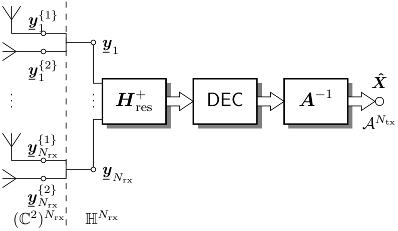
MIMO Multiple-Access Channel

- factorization $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{res}}\boldsymbol{A}$
- $\boldsymbol{H}_{\mathrm{res}} \in \mathbb{H}^{N_{\mathrm{rx}} \times N_{\mathrm{tx}}}$
- $\boldsymbol{A} \in \boldsymbol{\Lambda}_{\mathrm{a}}^{N_{\mathrm{tx}} \times N_{\mathrm{tx}}} = \mathbb{I}^{N_{\mathrm{tx}} \times N_{\mathrm{tx}}}$

  $\rightarrow$ quaternion-valued integer ring $\mathbb{I}$

  $\rightarrow$ how to factorize channel?



Quaternion-Valued LLL Reduction

- recapitulation
  - generalization of size-reduction condition: $\mathcal{Q}_{\mathbb{I}}\{r_{l,v}\} = 0$
  - maximum squared quantization error ϵ is minimum quality parameter $\Rightarrow \delta > \epsilon$
  - maximum quality parameter $\delta = 1$, i.e., $\delta \leq 1$
- LLL reduction over Lipschitz integers
  - maximum squared quantization error $\epsilon = 1$
  - however: $\epsilon = 1 < \delta \leq 1$

  $\Rightarrow$ LLL reduction over Lipschitz integers cannot be defined

  $\Rightarrow$ direct consequence of non-Euclidean property

Quaternion-Valued LLL Reduction

- LLL reduction over Hurwitz integers
  - maximum squared quantization error $\epsilon = 1/2$
  - hence: $1/2 < \delta \leq 1$

$\Rightarrow$ LLL reduction over Hurwitz integers can be defined

$\Rightarrow$ "QLLL reduction" [SF '18]

Quaternion-Valued Minkowski Reduction

- recapitulation
  - shortest vectors that can be extended to a basis of the lattice
  - condition: $\gcd\{\zeta_v, \ldots, \zeta_V\} = 1$

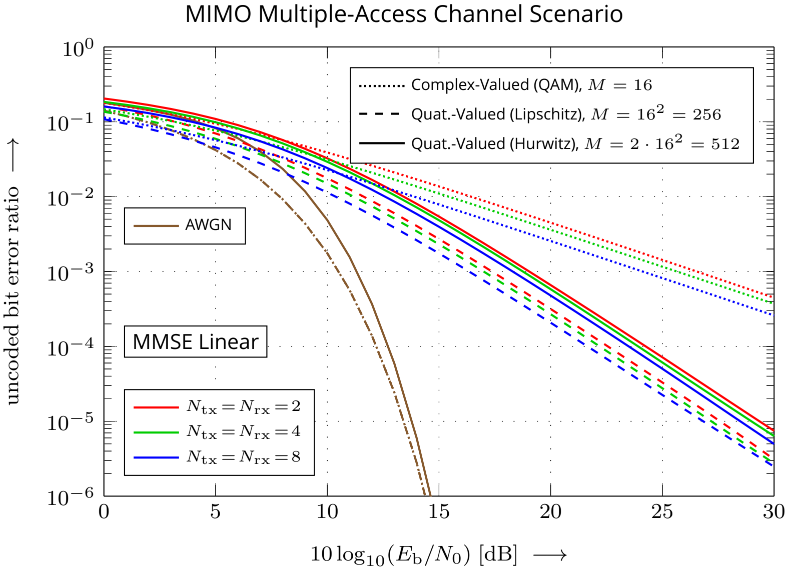
    $\rightarrow$ Euclidean algorithm required
- Minkowski reduction over Lipschitz integers
  - not a Euclidean ring
  - definition not possible
- Minkowski reduction over Hurwitz integers
  - Euclidean ring
  - definition possible

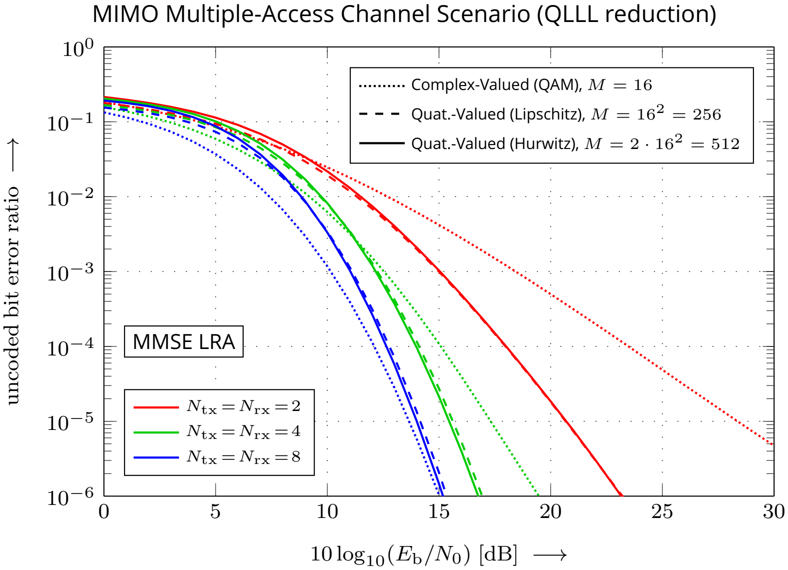
Quaternion-Valued Successive Minima Problem

- recapitulation: shortest linearly independent lattice vectors
- no restriction to Euclidean rings imposed
- possible over Lipschitz and Hurwitz integers

Diversity Order (i.i.d. Gaussian Channel)

- LRA linear equalization achieves full receive diversity [TMK '07]
- complex-valued transmission: $\Delta_{\mathrm{c,LRA}} = N_{\mathrm{rx}}$
- quaternion-valued transmission: $\Delta_{\mathrm{q,LRA}} = 2\,N_{\mathrm{rx}}$


MIMO Multiple-Access Channel Scenario
Complex-Valued (QAM), $M = 16$
Quat.-Valued (Lipschitz), $M = 16^2 = 256$
Quat.-Valued (Hurwitz), $M = 2 \cdot 16^2 = 512$
AWGN
MMSE Linear
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 2$
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 4$
$N_{\mathrm{tx}} = N_{\mathrm{rx}} = 8$
uncoded bit error ratio ⟶
$10 \log_{10}(E_{\mathrm{b}}/N_0)$ [dB] ⟶
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30


MIMO Multiple-Access Channel Scenario (QLLL reduction)
Complex-Valued (QAM), $M = 16$
Quat.-Valued (Lipschitz), $M = 16^2 = 256$
Quat.-Valued (Hurwitz), $M = 2 \cdot 16^2 = 512$
MMSE LRA
$N_{tx} = N_{rx} = 2$
$N_{tx} = N_{rx} = 4$
$N_{tx} = N_{rx} = 8$
uncoded bit error ratio $\longrightarrow$
$10 \log_{10}(E_b/N_0)$ [dB] $\longrightarrow$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30

Quaternions

- algebraic structure with four orthogonal components
- multiplication non-commutative
- related rings Lipschitz and Hurwitz integers

Quaternion-Valued Coded Modulation

- Hurwitz constellations benefit from denser packing
- coded modulation via two-stage encoding/decoding scheme

Quaternion-Valued MIMO Transmission

- diversity orders doubled w.r.t. complex-valued case
- LRA equalization via QLLL (based on Hurwitz integers)

Future Work

- quaternion-valued lattice-basis-reduction algorithms
- quaternion-valued algebraic signal constellations
- coded modulation for quaternion-valued MIMO transmission

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