The Generalized Sieve Kernel

The Algorithmic Ant and the Sandpile

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Based on joint work in progress with

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Lattice Coding Crypto Meeting

London, Sept 2018

[1]Supported by a Veni Innovational Research Grant from NWO (639.021.645).

The Algorithmic Ant and the Sandpile

Once upon a time . . .

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. . . there was an ant.

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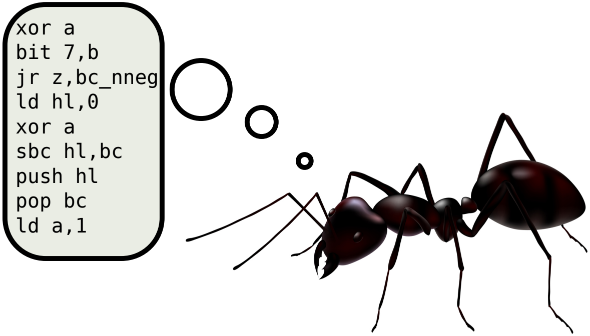
. . . there was an ant.



```
xor a
bit 7,b
jr z,bc_nneg
ld hl,0
xor a
sbc hl,bc
push hl
pop bc
ld a,1
```

An algorithmic ant.

The Queen of ant ant,

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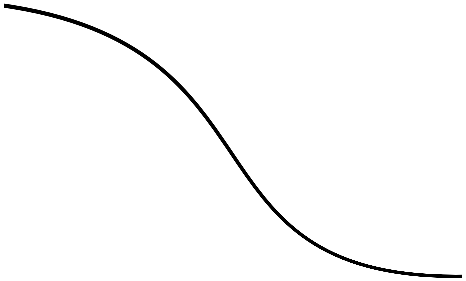
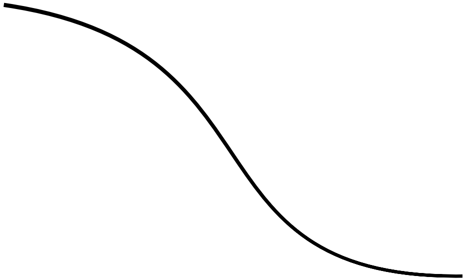
"See this sand pile."





"I want it *flat* !"

Looking clo

the algorithmic ant ponders.

"One grain at the time,
I shall pull the sand downhill."

"One grain at the time,
I shall pull the sand downhill."

So ho

But the sandpile is whimsical,
each excavation is a puzzle of it

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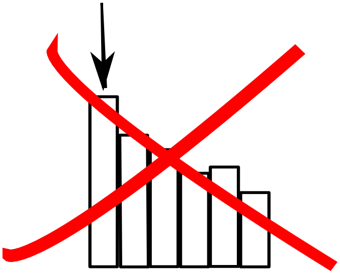

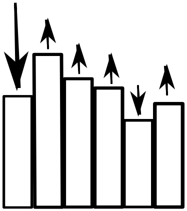

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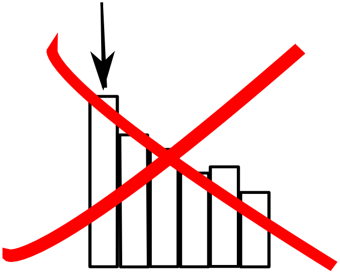
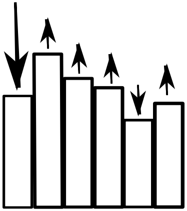
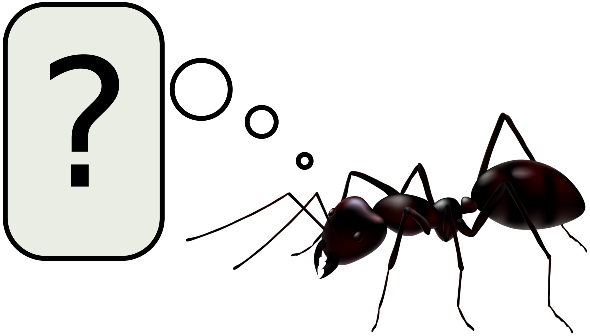




Columns are tied in my
to push one down, one must find
the right combination.

Unsure how to proceed,



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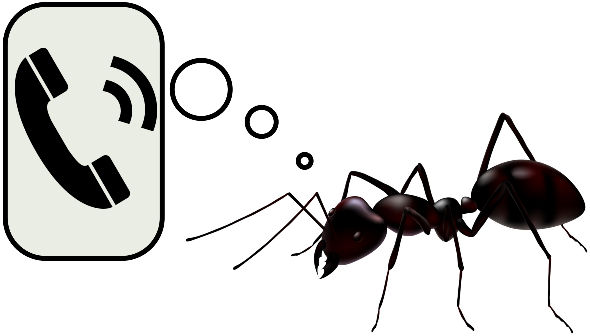


The ant call

"Let's break this apart."

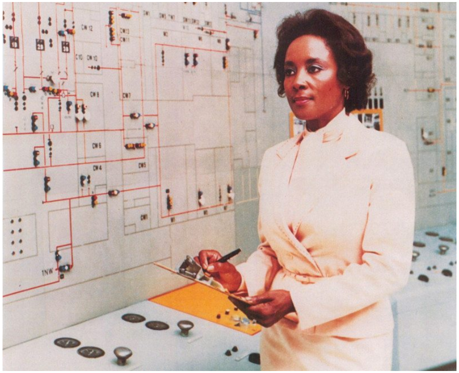
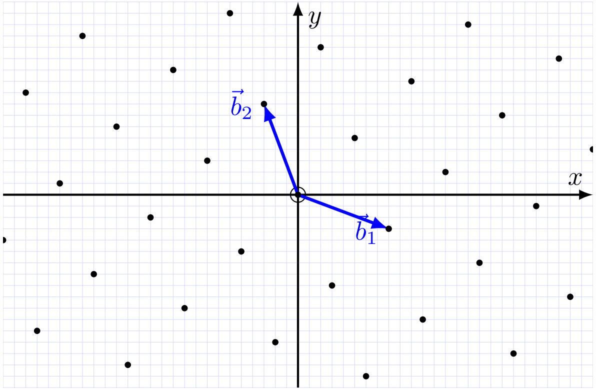


Figure: Annie Easley (NASA / NACA)

1. From Lattices to Sandpiles
2. Finding a grain of sand: Progress on SVP from Sieving
3. Flattening the Pile: Progress on lattice reduction from Sieving

From Lattices to Sandpiles

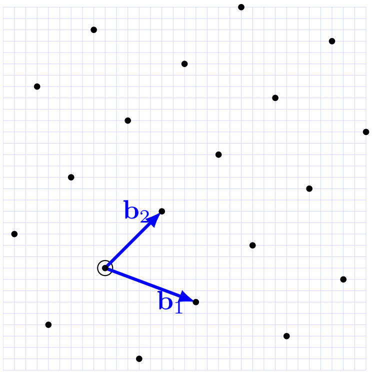
Lattices!



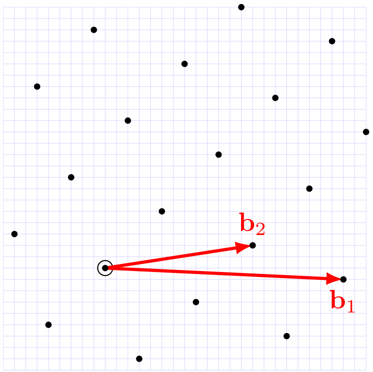
Definition

A lattice L is a discrete subgroup of a finite-dimensional Euclidean vector space.

Bases of a Lattice



Good Basis $\mathbf{G}$ of L



Bad Basis $\mathbf{B}$ of L

G $\rightarrow$ **B** : easy (randomization);
B $\rightarrow$ **G** : hard (LLL, BKZ, Lattice Sieve...).

An important invariant: the Volume

For any two bases $\mathbf{G}, \mathbf{B}$ of the same lattice Λ:

$$\det(\mathbf{G}\mathbf{G}^t) = \det(\mathbf{B}\mathbf{B}^t).$$

We can therefore define:

$$\mathrm{vol}(\Lambda) = \sqrt{\det(\mathbf{G}\mathbf{G}^t)}.$$

Geometrically: the volume of any **fundamental domain of** Λ.

Let $\mathbf{G}^\star$ be the Gram-Schmidt Orthogonalization of $\mathbf{G}$

$\mathbf{G}^\star$ is **not** a basis of Λ, nevertheless:

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What is a "Good" basis

Recall that, independently of the basis $\mathbf{G}$ it holds that:

$$\mathrm{vol}(\Lambda) = \prod \|\mathbf{g}_i^\star\|.$$

Therefore, it is somehow equivalent that

- $\max_i \|\mathbf{g}_i^\star\|$ is small
- $\min_i \|\mathbf{g}_i^\star\|$ is large
- $\kappa(\mathbf{G}) = \max_i \|\mathbf{g}_i^\star\| / \min_i \|\mathbf{g}_i^\star\|$ is small

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$$\max \|\mathbf{b}_i^*\| \approx \min \|\mathbf{b}_i^*\|$$

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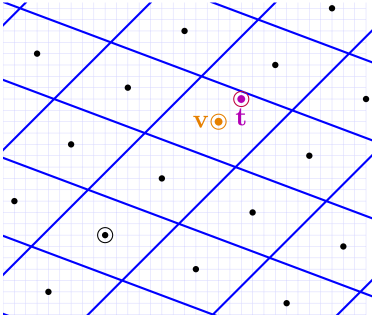
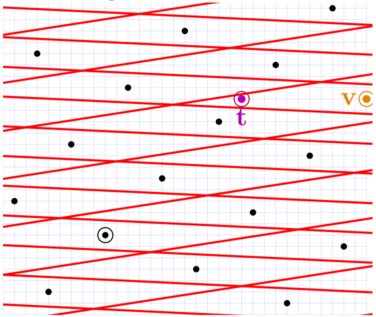
Bases and Fundamental Domains

Each basis defines a **parallelepipedic tiling**.





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- Given a target $\mathbf{t}$
- Find's $\mathbf{v} \in L$ at the center the tile.

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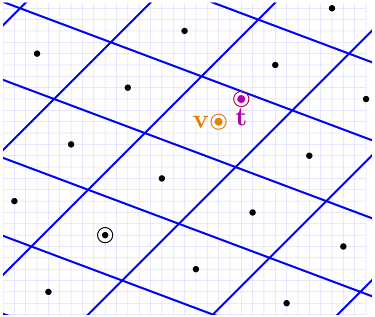

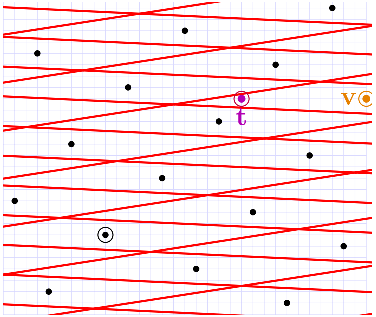

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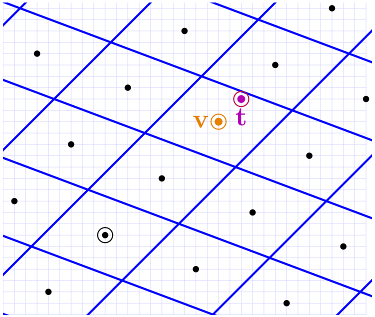

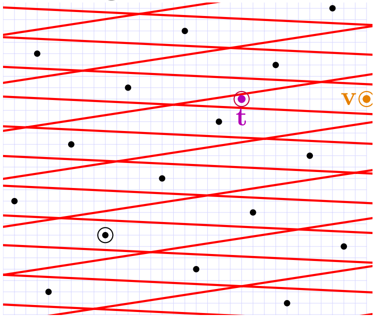

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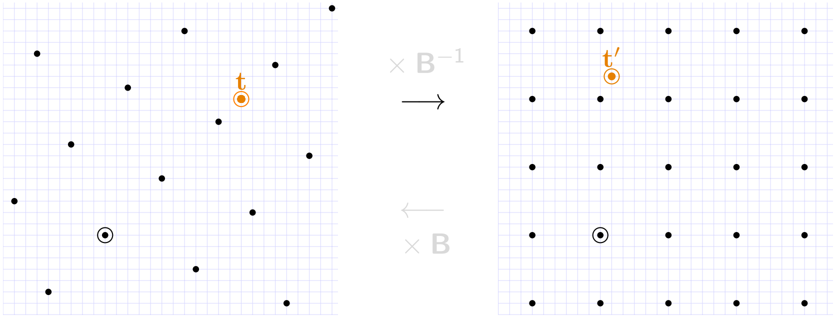

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- Use $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^n$ ($\times \mathbf{B}^{-1}$)
- round each coordinate (square tiling)
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$$\mathbf{t}' = \mathbf{B}^{-1} \cdot \mathbf{t}; \quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil; \quad \mathbf{v} = \mathbf{B} \cdot \mathbf{v}'$$

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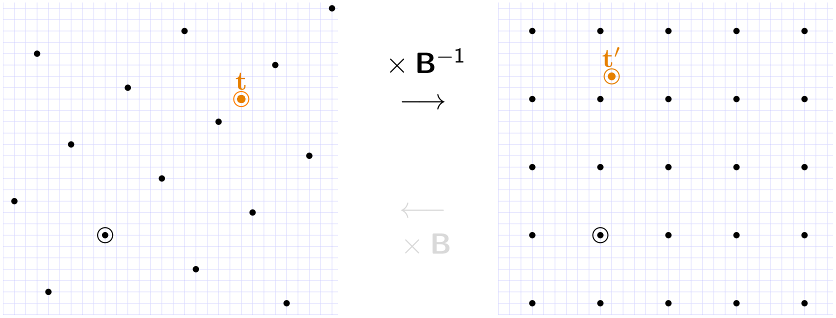

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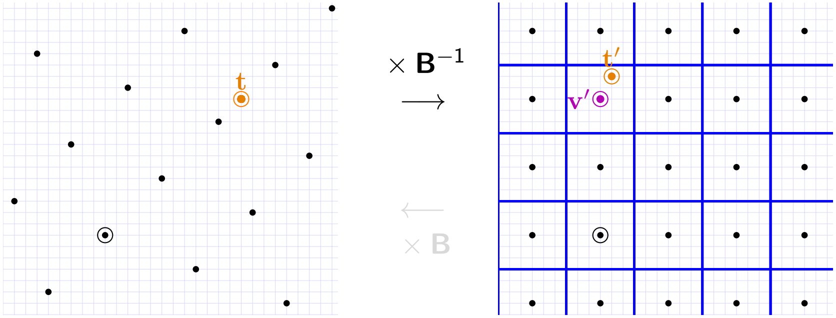

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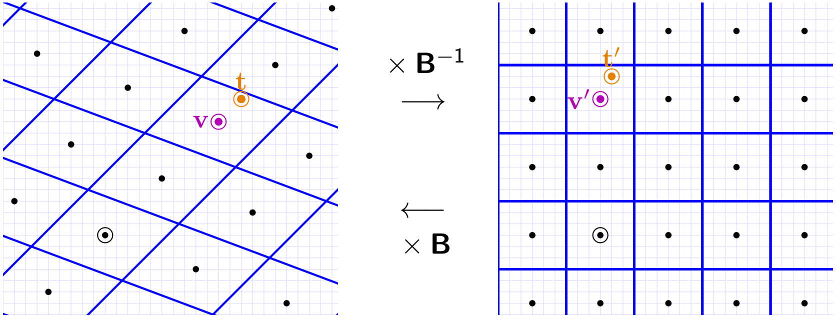

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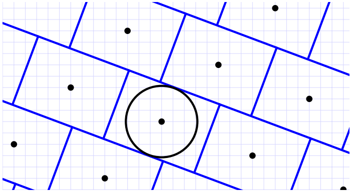
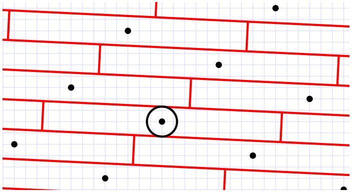
Nearest-Plane Algorithm

There is a better algorithm (NEARESTPLANE) based on Gram-Schmidt Orth. $\mathbf{B}^\star$ of a basis $\mathbf{B}$:


Decoding radius with $\mathbf{G}^\star$


Decoding radius with $\mathbf{B}^\star$

- Worst-case distance: $\frac{1}{2}\sqrt{\sum \|\mathbf{b}_i^\star\|^2}$ (Approx-CVP)
- Correct decoding of $\mathbf{t} = \mathbf{v} + \mathbf{e}$ where $\mathbf{v} \in \Lambda$ if (BDD)

$$\|\mathbf{e}\| \leq \frac{1}{2} \min \|\mathbf{b}_i^\star\|$$

Profile of a Basis

Good basis

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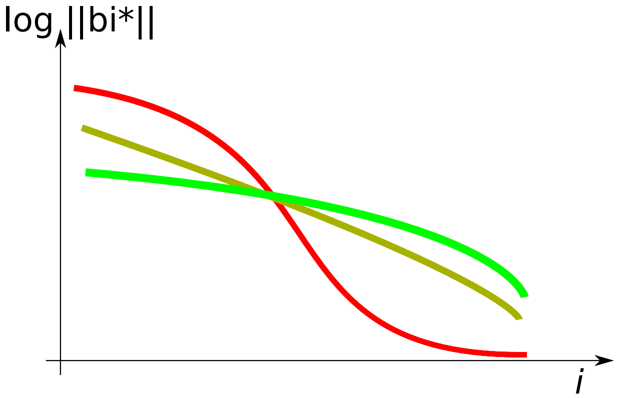

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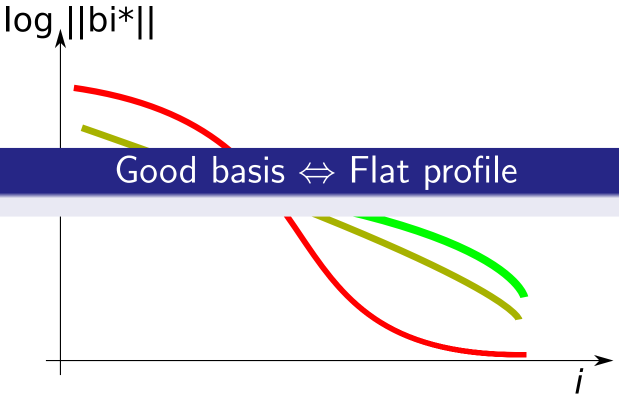
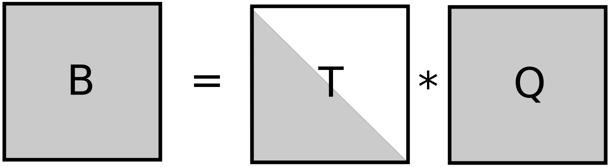


Good basis $\Leftrightarrow$ Flat profile

Local Modification



- Local blocks $[i:j]$ of T correspond to a projected sublattice $L_{[i:j]}$
- We can work locally: modify this block, affecting only $\mathbf{b}_i^* \dots \mathbf{b}_j^*$

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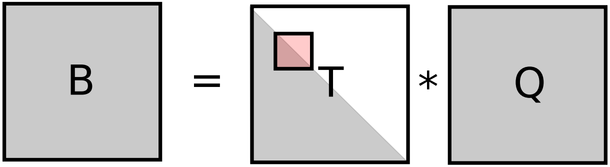

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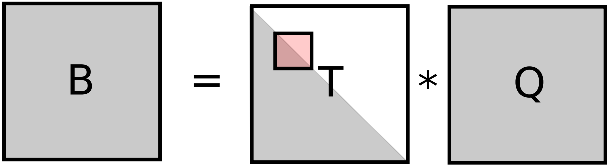

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Local Improvement

- Find the shortest vector v of the projected sublattice $L_{[i:j]}$

"a puzzle of it
"the right combination."

- Construct a unimodular matrix $\mathbf{U}$ such that $\mathbf{T}_{[i:j]} \cdot \mathbf{U} = [\mathbf{v}, *, *, \dots]$. Apply $\mathbf{U}$ (locally).
- The new $\mathbf{b}_i^* = v$ got shorter!
- The other $\mathbf{b}_{i+1}^*, \dots, \mathbf{b}_j^*$ will change as well

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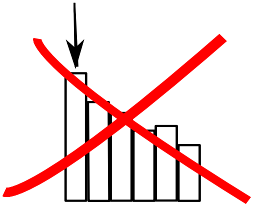

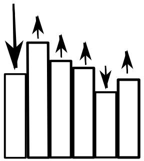

Lattice reduction (e.g. BKZ-b)

b: Blocksize
Run the local improvements for consecutive blocks:
$[1 : b], [2 : b+1], [3 : b+2], \ldots, [n-b : n], [n-b+1 : n], \ldots\ [n-1 : n]$
This is called a tour.
Repeat tours until satisfication (or convergence).

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BKZ in action

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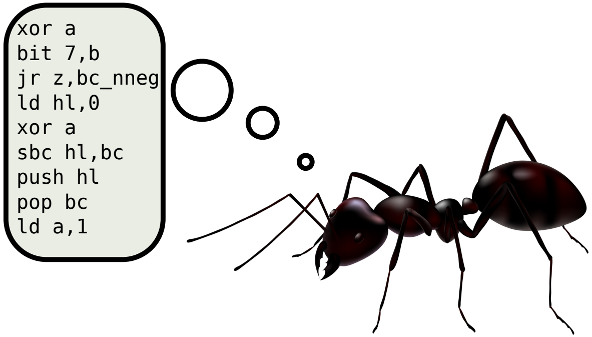

```
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```

how should I solve the SVP puzzle

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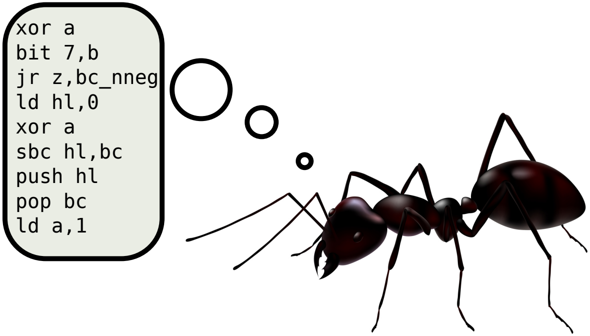

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how should I solve the *SVP* puzzle

Shortest Vector from Lattice Sieving: a Few Dimensions for Free[2]

[2] EUROCRYPT 2018

Two classes of Algorithms for SVP

The Shortest Vector Problem

I: The basis $\mathbf{B}$ of an n-dimensional lattice $\mathcal{L}$

O: A shortest non-zero vector $\mathbf{v} \in \mathcal{L}$

Algorithm	Running time	Memory
Enumeration	$n^{n/2e} \cdot 2^{O(n)}$	$\mathrm{poly}(n)$
Sieving[3]	$[2^{.292n+o(n)}, 2^{.415n+o(n)}]$	$[2^{.2075n+o(n)}, 2^{.292n+o(n)}]$

The paradox

In theory, Sieving is faster. In pratice it is quite a lot slower.

[3]Given complexities are heuristic, heavily supported by experiments.

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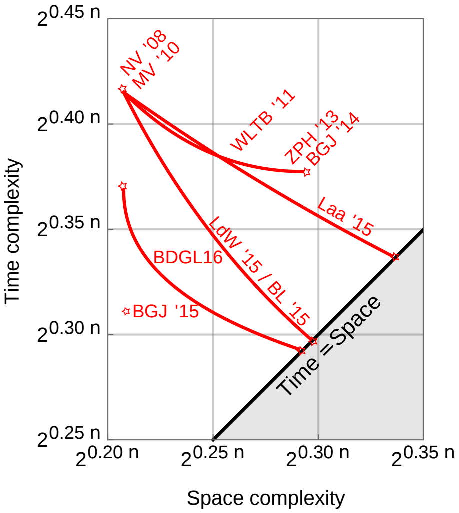

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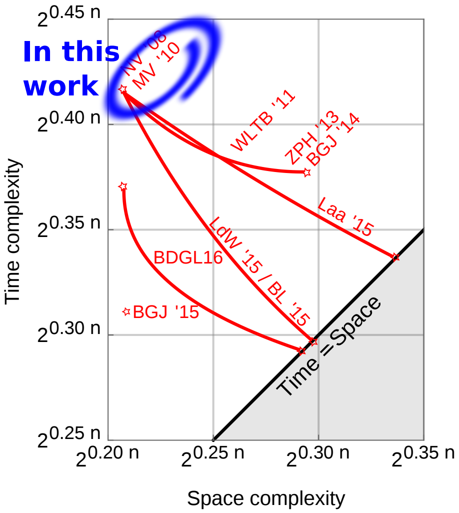

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Results

Heuristic claim, asymptotic

One can solve SVP in dimension n with a call to SIEVE in dimension $n - d$

where $d = \Theta(n/\log n)$.

Heuristic claim, concrete

One can solve SVP in dimension n making a call to SIEVE in dimension i for each $i = 2 \dots n - d$ for

$$d \approx \frac{n \cdot \ln(4/3)}{\ln(n/2\pi e)} \qquad (d \approx 15 \text{ for } n = 80)$$

Experimental claim: A bogey

A SIEVE implem. almost on par with enumeration (within a factor 4 in dims 70–80), still with room for many improvements.

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Sieving

Algorithm 1 SIEVE($\mathcal{L}$)

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$L \leftarrow$ a set of $N$ random vectors from $\mathcal{L}$ where $N \approx (4/3)^{n/2}$.
while $\exists(\mathbf{v}, \mathbf{w}) \in L^2$ such that $\|\mathbf{v} - \mathbf{w}\| < \|\mathbf{v}\|$ do
    $\mathbf{v} \leftarrow \mathbf{v} - \mathbf{w}$
end while
return $L$
```

The above runs in heuristic time $(4/3)^{n+o(n)}$.

Many concrete and asymptotic improvements:
[Nguyen Vidick 2008, Micciancio Voulgaris 2010, Laarhoven 2015, Becker Gamma Joux 2015, Becker D. Gamma Laarhoven 2015, . . .].

Sieving

Algorithm 2 $\textsc{Sieve}(\mathcal{L})$

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More than SVP

Note that $\textsc{Sieve}$ returns $N \approx (4/3)^n$ short vectors, not just a shortest vector.

Definition (Gaussian Heuristic: Expected length of the shortest vector)

$$\mathrm{gh}(\mathcal{L}) = \sqrt{n/2\pi e} \cdot \mathrm{vol}(\mathcal{L})^{1/n}.$$

Observation (heuristic & experimental)

The output of $\textsc{Sieve}$ contains almost all vectors of length $\leq \sqrt{4/3} \cdot \mathrm{gh}(\mathcal{L})$:

$$L := \textsc{Sieve}(\mathcal{L}) = \left\{ \mathbf{x} \in \mathcal{L} \text{ s.t. } \|\mathbf{x}\| \leq \sqrt{4/3} \cdot \mathrm{gh}(\mathcal{L}) \right\}.$$

Sieve then Lift

Main idea: Sieve in a projected sub-lattice, and lift all candidate solutions.

$\textsc{SubSieve}(\mathcal{L}, d)$

- Set $\mathcal{L}' = \mathcal{L}(\mathbf{b}_1, \ldots, \mathbf{b}_d)$ "left part of $\mathcal{L}$", dim=d
- Set $\mathcal{L}'' = \pi^{\perp}_{\mathcal{L}'}(\mathcal{L})$ "right part of $\mathcal{L}$", dim=$n-d$
- Compute $L = \textsc{Sieve}(\mathcal{L}'')$
- Hope that $\pi^{\perp}_{\mathcal{L}'}(\mathbf{s}) \in L$ (1)
- Lift all $\mathbf{v} \in L$ from $\mathcal{L}''$ to $\mathcal{L}$ and take the shortest (Babai alg.)

Pessimistic prediction for (1)

$$\mathrm{gh}(\mathcal{L}) \leq \sqrt{4/3} \cdot \mathrm{gh}(\mathcal{L}'').$$

Optimistic prediction for (1)

$$\sqrt{\frac{n-d}{n}} \cdot \mathrm{gh}(\mathcal{L}) \leq \sqrt{4/3} \cdot \mathrm{gh}(\mathcal{L}'').$$

Similar to linear pruning for enum.

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With BKZ pre-processing

- To ensure (1), we need the basis to be as reduced as possible
- We can easily afford BKZ preprocessing with block-size $b = n/2$
- Using simple BKZ models[4] we can predict $\mathrm{gh}(\mathcal{L})$ and $\mathrm{gh}(\mathcal{L}')$

Heuristic claim

$\textsc{SubSieve}(\mathcal{L}, d)$ algorithm will successfully find the shortest vector of $\mathcal{L}$ for some $d = \Theta(n/\ln n)$.

$\Rightarrow$ Improve time & memory by a sub-exponential factor $2^{\Theta(n/\log n)}$

[4] The Geometric Series Assumption

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[4]The Geometric Series Assumption

Quasi-HKZ preprocessing

Idea: Attempt stronger pre-processing.

Algorithm 3 $\textsc{SubSieve}^+(\mathcal{L}, d)$

```
$L \leftarrow \textsc{Sieve}(\mathcal{L}'')$
$L = \{\textsc{Lift}_{\mathcal{L}'' \to \mathcal{L}}(v) \text{ for } v \in L\}$
for $j = 0 \dots n/2 - 1$ do
    $\mathbf{v}_j = \arg\min_{\mathbf{s} \in L} \|\pi_{(\mathbf{v}_0 \dots \mathbf{v}_{j-1})^\perp}(\mathbf{s})\|$
end for
return $(\mathbf{v}_0 \dots \mathbf{v}_{n/2-1})$
```

- Insert $(\mathbf{v}_0 \dots \mathbf{v}_{n/2-1})$ as the new $\mathbf{b}_1 \dots \mathbf{b}_{n/2}$
- Repeat $\textsc{SubSieve}^+(\mathcal{L}, d)$ for $d = n-1, n-2, \dots, d_{\min}$
- Hope that iteration $d_{\min} + 1$ provided a quasi-HKZ basis.

Concrete prediction with quasi-HKZ preprocessing

Pessimistic prediction for (1)

$$d \approx \frac{n \ln 4/3}{\ln(n/2\pi)}$$

Optimistic prediction for (1)

$$d \approx \frac{n \ln 4/3}{\ln(n/2\pi e)}$$

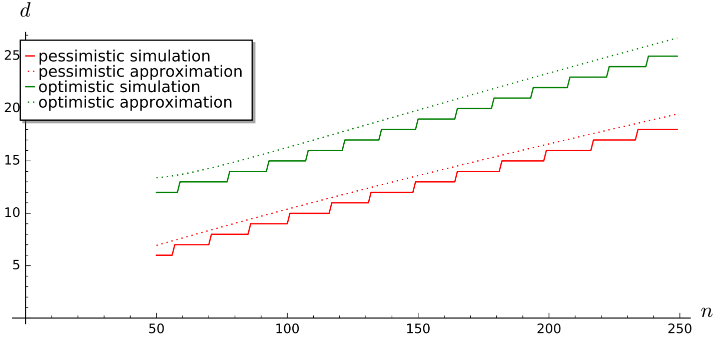

Figure: Predictions of the maximal successful choice of d_{min}.

Baseline Implementation (V0)

Re-implemented GaussSieve **[Micciancio Voulgaris 2010]**

- No gaussian sampling
  - Initial sphericity of L doesn't seem to matter
  - Initial vectors can be made much shorter $\Rightarrow$ speed-up
- Prevent collisions using a hash table
- Terminate when the ball $\sqrt{4/3} \cdot \mathrm{gh}(\mathcal{L})$ is half-saturated
- Sort only periodically
  - Can use faster data-structures
- Vectors represented in bases $\mathbf{B}$ and GramSchmidt($\mathbf{B}$)
  - Required to work in projected-sublattices
- Kernel in `c++`, control in `python`
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XOR−POPCNT trick (V0 $\to$ V1)

Already used in Sieving **[Fitzpatrick et al. 2015]**.
More generally know as SimHash **[Charikar 2002]**.

Idea: Pre-filter pairs $(\mathbf{v}, \mathbf{w}) \in L$ with a fast compressed test.

- Choose a spherical code $\mathcal{C} = \{\mathbf{c}_1 \dots \mathbf{c}_k\} \subset \mathcal{S}^n$ and a threshold $t \leq k/2$
- Precompute compressions $\tilde{\mathbf{v}} = \textsc{Sign}(\langle \mathbf{v}, \mathbf{c}_i \rangle) \in \{0,1\}^k$
- Only test $\|\mathbf{v} \pm \mathbf{w}\| \leq \|\mathbf{v}\|$ if

$$|\textsc{HammingWeight}(\mathbf{v} \oplus \mathbf{w}) - k/2| \geq t.$$

- Asymptotic speed-up $\Theta(n/\log n)$?
- In practice, $k = 128$ (2 words), $t = 18$: about 10 cycles per pairs.

Progressive Sieving (V1 $\rightarrow$ V2)

Concurrently and independetly invented in **[Mariano Laarhoven 2018]**.

Idea: Increase the dimension progressively.

- Recursively, Sieve in the lattice $\mathcal{L}(\mathbf{b}_1, \ldots \mathbf{b}_{n-1})$
- Start the sieve in dimension n with many short-ish vectors
- Fresh vectors get reduced much faster thanks to this initial pool.

Refer to **[Mariano Laarhoven 2018]** for a full analysis of this trick.

Dimensions for Free (V2 $\rightarrow$ V3)

- Apply the quasi-HKZ preprocessing strategy
- Do not force the choice of $d_{\min}$
- Simply increase d until the shortest vector is found.

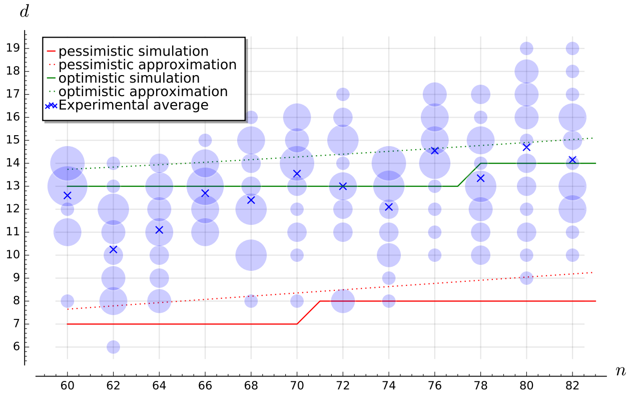
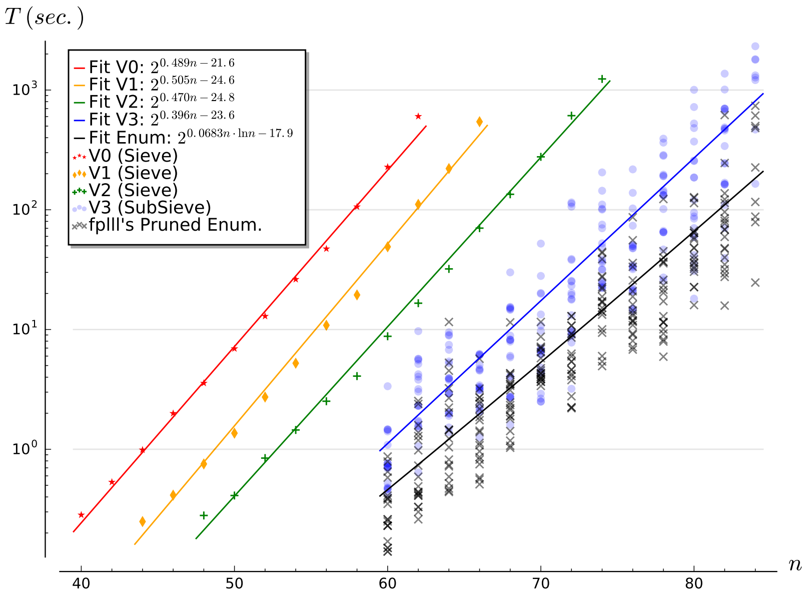


Figure: Predictions experiments for $d_{\min}$.

Performances



Comparison to other Sieving implementation

	Algorithms							
	V0	V1	V2	V3	[MV10]	[FBB+14]	[ML17]	[HK17]
Features								
`XOR-POPCNT` trick		x	x	x		x		
pogressive sieving			x	x				
SubSieve				x				
LSH (more mem.)							x	
tuple (less mem.)								x
Dimension	Running times							
$n = 60$	227s	49s	8s	0.9s	464s	79s	13s	1080s
$n = 70$	-	-	276s	10s	23933s	4500s	250s	33000s
$n = 80$	-	-	-	234s	-	-	4320s	94700s
CPU freq. (GHz)	3.6	3.6	3.6	3.6	4.0	4.0	2.3	2.3

Summary

Sieving vs. Sieving

- Exploit all outputs of Sieve $\Rightarrow$ Dimensions for Free
- Our implementation is 10x faster than all previous Sieving
- It does not use LSH techniques: further speed-up expected

Sieving vs. Enumeration

- Only a factor 4x slower than Enum for dimensions 70–80
- Guesstimates a cross-over at dim $\approx$ 90 with further improvements (LSH/LSF, fine-tuning, vectorization, ...)

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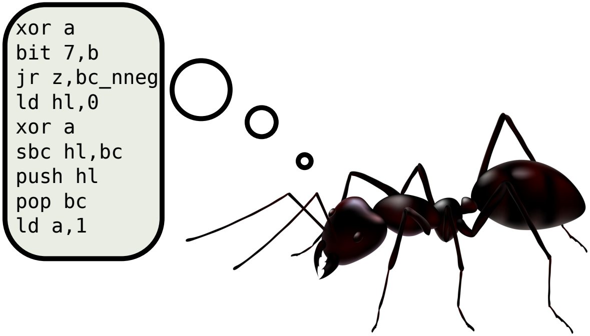

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xor a
bit 7,b
jr z,bc_nneg
ld hl,0
xor a
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push hl
pop bc
ld a,1
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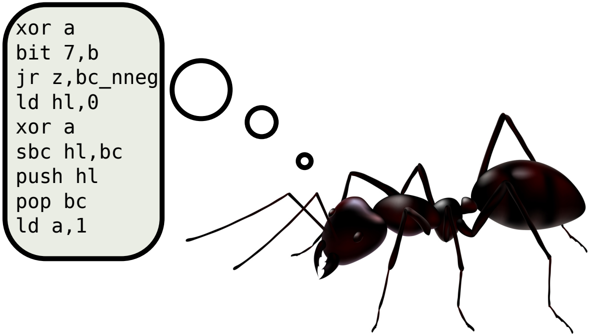

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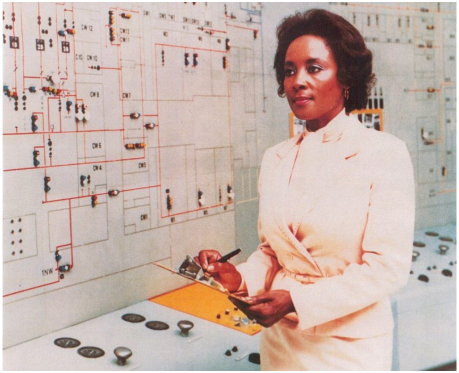

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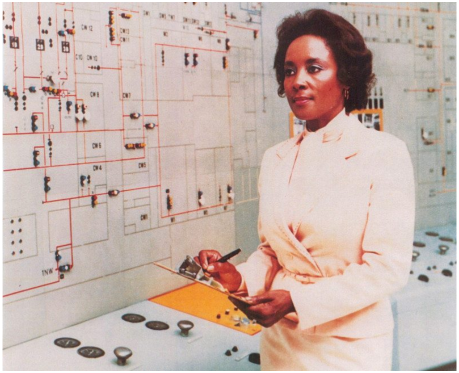

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The Generalized Sieve Kernel
(G6K, pronounced /ζe.si.ka/) [5]

[5]Work in Progress with M. Albrecht, E. Postlethwaite, G. Herold, E. Kirshanova, M. Stevens

1st design principle: Go Green !

Idea: Recycle vectors between overlapping blocks.

Rather than an function serving as an SVP oracle, design a **stateful machine** that takes advantages of the overlapping instances.

In other words:

An Algorithmic Ant on a Sandpile, carrying a bag of vectors on it

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Moving with a bag of vectors

Relations between the projected sublattices:

$$\begin{tikzcd}
L = & L_{[1:n]} \arrow[d, "\pi"] & \supset & L_{[1:n-1]} \arrow[d, "\pi"] & \supset & \cdots & L_{[1:2]} \arrow[d, "\pi"] & \supset & \ldots L_{[1:1]} \\
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\end{tikzcd}$$

Change of context/block `[l:r]` : transform the vectors in the bag

- Extend-Right : $\subset$ (do nothing)
- Shrink-Left : π^{-1} (Babai lift)
- Extend-Left : π (project)

2nd design principle: be flexible

BKZ theory use exact-SVP for each block consecutively, but maybe we're better off making different choices.

- Maintain a cadidate for insertion at **each** position
- Decide where to insert **after** sieving

3rd design principle: seize opportunities

Algorithm 4 $\textsc{Sieve}(\mathcal{L})$

```
$L \leftarrow$ a set of $N$ random vectors from $\mathcal{L}$ where $N \approx (4/3)^{n/2}$.
while $\exists(\mathbf{v}, \mathbf{w}) \in L^2$ such that $\|\mathbf{v} - \mathbf{w}\| < \|\mathbf{v}\|$ do
    $\mathbf{v} \leftarrow \mathbf{v} - \mathbf{w}$
end while
return $L$
```

Even if $\|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{v}\|$, it could be worth considering the lifts of $\mathbf{v} - \mathbf{w}$.

The abstract machine

State:

- A lattice basis $\mathbf{B}$
- Positions $0 \leq \ell' \leq \ell \leq r \leq d$.
  $[\ell : r]$ the *sieving context*, and $[\ell' : r]$ the *lifting context*.
- A database *db* of N vectors in $\mathcal{L}_{[\ell:r]}$ (preferably short).
- Insertion candidates $\mathbf{c}_{\ell'}, \dots, \mathbf{c}_{\ell}$ where $\mathbf{c}_i \in \mathcal{L}_{[i:r]}$ or $\mathbf{c}_i = \perp$.

Instructions:

- Sieve (`S`): make vector shorter, improve insertion candidates
- Extend Right, Shrink Left, Extend Left (`ER`, `SL`, `EL`): change the sieve-context, updating the database
- Insert (`I`): update the basis and the database

The ideal BKZ with G6K

BKZ can be written very simply:

`Repeat {S; I; ER; }`

When starting the second `Sieve`, vectors are already quite short
$\Rightarrow$ No need to restart progressive sieving from the beginning.

The ER bug.

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Pump

Before:

$$\texttt{SubSieve}_f : \mathit{Reset}_{0,f,f},\ (\texttt{ER},\ \texttt{S})^{d-f},\ \texttt{I}_0, \texttt{I}_1, \ldots, \texttt{I}_{d-f}.$$

- No issues with EL $\Rightarrow$ Progressive-Sieving toward the left instead.
- Can now Sieve again after insertion
- Can now insert the best candidate rather than a pre-chosen one

$$\texttt{Pump}_{\ell',\ell,r,s} :\ \texttt{Reset}_{\ell',r,r},\ \overbrace{(\texttt{EL},\ \texttt{S})^{r-\ell}}^{\text{pump-up}},\ \overbrace{(\texttt{I},\ \texttt{S})^{r-\ell}}^{\text{pump-down}}.$$

WorkOut

Workout: Pumps of increasing strength

$$\begin{aligned} \texttt{WorkOut}_{\kappa,\beta,f,f^+,s} \ : \ & \texttt{Pump}_{\kappa,\kappa+\beta-f^+,\kappa+\beta,s}, \\ & \texttt{Pump}_{\kappa,\kappa+\beta-2f^+,\kappa+\beta,s}, \\ & \texttt{Pump}_{\kappa,\kappa+\beta-3f^+,\kappa+\beta,s}, \\ & \ldots \\ & \texttt{Pump}_{\kappa,\kappa+f,\kappa+\beta,s}, \end{aligned}$$

- Termination condition can vary (e.g. fixed number of dims for free, or reached satisfying shortest vector)
- steps size of pump strength is not necessarly 1

Pump and Jump

- Block is left somewhat reduced by the pump in the previous block: $\Rightarrow$ no need for a full workout.
- Many short vectors inserted, little improvement left around here: $\Rightarrow$ directly Jump far away.

$$\mathtt{PumpnJumpBKZ}_{\beta',f,j} : \mathtt{Pump}_{0,f,\beta},\ \mathtt{Pump}_{j,j+f,j+\beta},\ \mathtt{Pump}_{2j,2j+f,2j+\beta},\dots$$

Implementation

3 layers

- c++: multi-threaded heavy duty operation (Sieve, *db* updates)
- cython: middleware, basis maintainance
- python: control, tuning, and monitoring

Several Sieve inside:

- Standard Gauss-Sieve (mono-threaded)

$$Mem = 2^{.208n+o(n)}, Time = 2^{.415n+o(n)}$$

- Becker-Gama-Joux with 1 level of filtration (multi-threaded)

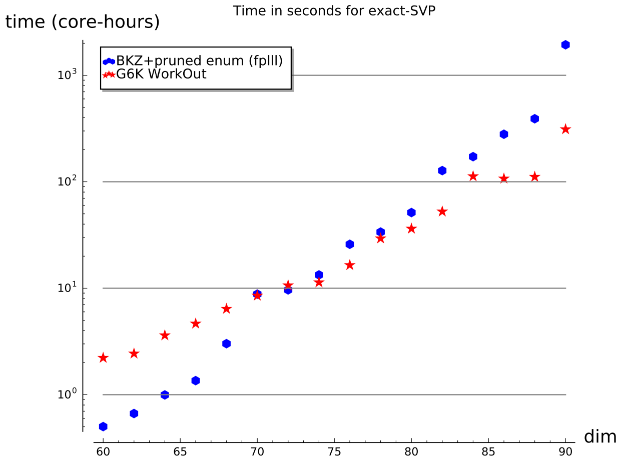
$$Mem = 2^{.208n+o(n)}, Time = 2^{.349n+o(n)}$$

- k-sieve $k = 2, 3$ (multi-threaded)

$$Mem = 2^{.208n+o(n)}, Time = 2^{.349n+o(n)}$$

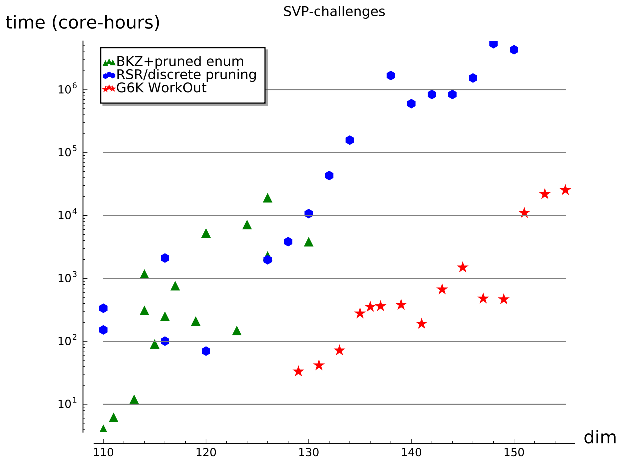
$$Mem = 2^{.189n+o(n)}, Time = 2^{.372n+o(n)}$$

Performances: Exact-SVP



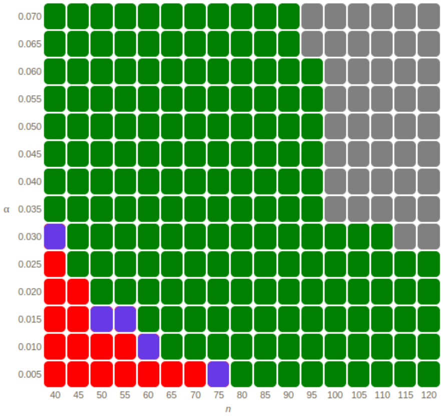
- About 4 extra dims for free
- Cross-over with enum at dim ≈ 70

Records: SVP-challenges



- Solved challenges up to dim 155, with 80 cores in 14 days
- About 400x faster than previous records

Records: LWE-challenges



Red: solved (prior) Blue: solved (ours) Green: unsolved.

- New cost-balancing trick improving upon the prediction of [AGVW17]

Stay tuned

- Paper to be finalized
- Implementation will be made open-source

Thanks!