Parameter Selection in Ring-LWE-based Fully Homomorphic Encryption

Rachel Player
Information Security Group, Royal Holloway, University of London

based on joint works with
Martin R. Albrecht, Hao Chen, Kim Laine, Sam Scott, and Yuhou Xia

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Setting the scene

Lattice-based crypto:
- Candidate for post-quantum crypto
- Parameter selection is a drawback
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Fully Homomorphic Encryption:
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Fully Homomorphic Encryption:
- the coolest application of lattice-based crypto
Setting the scene

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Fully Homomorphic Encryption:
- the coolest application of lattice-based crypto
- an interesting setting for parameter selection
What is homomorphic encryption?

\[ F(x) \]

Encryption of \( x \)

\[ \text{Eval}(x, F) \]

Encryption of \( F(x) \)
Achieving homomorphic encryption

- Homomorphic addition
- Homomorphic multiplication
Applications of homomorphic encryption

- Healthcare
- Genomics
- Private set intersection
- Signal processing
- Machine learning
- ...
Is homomorphic encryption practical?

- First schemes very impractical
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- Many implementations now exist:
  - HElib
  - SEAL
  - FV-NFLlib, Palisade, HEAAN, cuHE, TFHE, ...
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- Standardisation effort:
  [https://homomorphicencryption.org](https://homomorphicencryption.org)
Is homomorphic encryption practical?

- First schemes very impractical
- Many implementations now exist:
  - HElib
  - SEAL
  - FV-NFLlib, Palisade, HEAAN, cuHE, TFHE, …
- Standardisation effort:
  - https://homomorphicencryption.org
- Results for specific applications
Learning with Errors (LWE) [R05]

- **Search**: given $A$ and $b$, recover $s$
- **Decision**: distinguish whether $(A, b)$ is chosen as LWE or uniformly at random
The ring $R_q$

Let $n$ be a power of 2 and define

$$R_q = \mathbb{Z}_q[x]/(x^n + 1)$$
The ring $R_q$

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$$R_q = \mathbb{Z}_q[x]/(x^n + 1)$$

Ring LWE (Decision)

Let $s \in R_q$ be a secret. Let $a \leftarrow R_q$ be chosen uniformly at random. Let $\chi$ be a distribution over $R_q$. Let $e \leftarrow \chi$. Distinguish $(a, b = as + e) \in R_q \times R_q$ from uniformly random $(a, b) \in R_q \times R_q$. 
Why is \( n \) a power of two?

Theorem [LPR12]

There is a polynomial time quantum reduction from approximate SIVP (Shortest Independent Vector Problem) on ideal lattices in \( K \) to Decision Ring-LWE in \( R \) given a fixed number of samples, where the error distribution is a fixed spherical Gaussian over the field tensor product \( K_R = K \otimes_{\mathbb{Q}} \mathbb{R} \).

If \( n = 2^k \):
- easy to implement
- performance benefit
What are the parameters?

A (Ring) LWE instance is specified by:

- $n$ dimension
- $q$ modulus
- $\alpha$ error distribution

where the standard deviation $\sigma$ of $\chi$ satisfies

$$\sigma = \frac{\alpha q}{\sqrt{2\pi}}$$
Is my Ring-LWE-based scheme secure?

- Parameters \( n, q, \alpha \) in the scheme imply an underlying Ring LWE instance
- Treat Ring LWE instance as an LWE instance
- Observe that LWE instance is hard to solve
LWE based FHE parameters are atypical

Typical LWE parameters (Regev)

- $q$ polynomial in $n$
- $\alpha q = \sqrt{n}$
LWE based FHE parameters are atypical

Typical LWE parameters (Regev)

- $q$ polynomial in $n$
- $\alpha q = \sqrt{n}$

FHE parameters

- huge $q$
- tiny error distribution e.g. $\alpha q = 8$
- small secret $\|s\| = 1$
- possibly sparse secret
So how hard is (small secret) LWE, anyway?

Theory

LWE with binary secret in dimension \( n \log q \) is as hard as general LWE in dimension \( n \). [BLP+13,MP13]

- Many approaches for solving LWE
- Even more in the case of small and/or sparse secret
[APS15] estimator for hardness of LWE instances

https://bitbucket.org/malb/lwe-estimator

**input** LWE instance \( n, q, \alpha \)

**output** estimates of runtime, memory, samples

Can optionally specify:
- Limited samples [BBGS17]
- Secret distribution
- Lattice reduction cost method
Running example: SEAL [DGBL+15,LP16,CLP16,CLP17]

- Homomorphic encryption library
- Developed by Microsoft Research
- Current version v2.2, June 2017
- Implements FV scheme [FV12]

sealcrypto.org
FV is IND-CPA secure if Ring LWE is hard

**SecretKeyGen**: Output $s \leftarrow R_2$

**PublicKeyGen**: Sample $a \leftarrow R_q$, and $e \leftarrow \chi$. Output

$$(p_0, p_1) = ([-(as + e)]_q, a)$$

**Encrypt**($\langle p_0, p_1 \rangle, m$): Sample $u \leftarrow R_2$, and $e_1, e_2 \leftarrow \chi$. Output

$$(c_0, c_1) = ([\Delta m + p_0 u + e_1]_q, [p_1 u + e_2]_q)$$
Choosing SEAL parameters for security

Already fixed are

- $n$ a power of two
- $\sigma = 3.2$
- some threshold $\lambda$
Choosing SEAL parameters for security

Already fixed are

- $n$ a power of two
- $\sigma = 3.2$
- some threshold $\lambda$

Find an acceptable bit length of $q$

- Choose initial bit length $K$
- Use [APS15] estimator to determine best attack for $n, q = 2^K, \alpha = 8/q$
- If best attack costs less than $\lambda$, decrement $K$ and repeat
- If best attack costs more than $\lambda$, stop
Estimate of SEAL v2.2 security [CLP17]

<table>
<thead>
<tr>
<th>n</th>
<th>q</th>
<th>$\alpha$</th>
<th>usvp</th>
<th>dec</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>2048</td>
<td>$2^{60} - 2^{14} + 1$</td>
<td>$8/q$</td>
<td>115.5</td>
<td>127.1</td>
<td>118.4</td>
</tr>
<tr>
<td>4096</td>
<td>$2^{116} - 2^{18} + 1$</td>
<td>$8/q$</td>
<td>119.7</td>
<td>125.3</td>
<td>121.2</td>
</tr>
<tr>
<td>8192</td>
<td>$2^{226} - 2^{26} + 1$</td>
<td>$8/q$</td>
<td>123.6</td>
<td>126.3</td>
<td>124.0</td>
</tr>
<tr>
<td>16384</td>
<td>$2^{435} - 2^{33} + 1$</td>
<td>$8/q$</td>
<td>129.5</td>
<td>130.7</td>
<td>130.2</td>
</tr>
<tr>
<td>32768</td>
<td>$2^{889} - 2^{54} - 2^{53} - 2^{52} + 1$</td>
<td>$8/q$</td>
<td>127.3</td>
<td>127.6</td>
<td>127.4</td>
</tr>
</tbody>
</table>

Table: Estimates of the cost of solving LWE instances underlying SEAL v2.2 default parameters. Obtained using commit cc5f6e8 of the estimator in [APS15].
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Noise and correctness

- FHE ciphertexts all have noise
- Noise grows with homomorphic operations
- If noise too large, decryption will fail
Noise and correctness

- FHE ciphertexts all have noise
- Noise grows with homomorphic operations
- If noise too large, decryption will fail

The better our understanding of noise the easier it is to choose parameters, and we may be able to choose smaller parameters.
The FV scheme [FV12]

**SecretKeyGen:** Output $s \leftarrow R_2$

**PublicKeyGen($s$):** Sample $a \leftarrow R_q$, and $e \leftarrow \chi$. Output

$$(p_0, p_1) = ([-(as + e)]_q, a)$$

**Encrypt($($p_0$, $p_1$), $m$):** Sample $u \leftarrow R_2$, and $e_1, e_2 \leftarrow \chi$. Output

$$(c_0, c_1) = ([\Delta m + p_0 u + e_1]_q, [p_1 u + e_2]_q)$$

**Decrypt($s$, ($c_0$, $c_1$)):** Output

$$\left[\frac{t}{q} [c_0 + c_1 s]_q \right]_t$$
Existing notions of noise in FV

Inherent Noise \([FV12,CLP16]\)

- The inherent noise is \(v_{inh}\) such that \([c_0 + c_1 s]_q = \Delta m + v_{inh}\).
- We require \(\|v_{inh}\|_\infty < \frac{q}{2t} - \frac{t}{2}\)

Critical quantity \([CS16]\)

- The critical quantity is \(v_{inh} - \frac{r_t(q)}{t} m\)
- We require \(\|v_{inh} - \frac{r_t(q)}{t} m\|_\infty < \frac{\Delta}{2}\)
We want noise to be the thing which causes decryption to fail if it is too large.

Recall FV Decryption: $m \mod t = \left\lfloor \frac{t}{q}[c_0 + c_1s]_q \right\rfloor_t$

**Invariant noise [CLP17]**

$$\frac{t}{q}(c_0 + c_1s) = m + v + at$$

The norm $\|v\|_\infty$ is the **invariant noise**.
Invariant noise

By definition

$$\left\lfloor \frac{t}{q} [c_0 + c_1 s]_q \right\rfloor = m + \lfloor v \rfloor + a't$$

So FV decryption succeeds

$$\left\lfloor \frac{t}{q} [c_0 + c_1 s]_q \right\rfloor_t = m \mod t \quad \text{if} \quad \|v\|_\infty < \frac{1}{2}$$
Invariant noise

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So FV decryption succeeds

$$\left\lfloor \frac{t}{q} [c_0 + c_1 s]_q \right\rfloor_t = m \mod t \quad \text{if } \|v\|_\infty < \frac{1}{2}$$

Noise budget

Initial noise in a fresh ciphertext is very small, and even in later ciphertexts we have $2\|v\|_\infty < 1$ if decryption succeeds. Easier to work with the noise budget defined as $-\log_2 (2\|v\|)_\infty$. 
Homomorphic operations in SEAL

Addition \((ct_0, ct_1)\): Output

\((ct_0[0] + ct_1[0], ct_0[1] + ct_1[1])\)

Multiplication \((ct_0, ct_1)\): Compute

\[
c_0 = \left\lceil \frac{t}{q} \frac{ct_0[0]ct_1[0]}{} \right\rceil_q
\]

\[
c_1 = \left\lceil \frac{t}{q} \left( ct_0[0]ct_1[1] + ct_0[1]ct_1[0] \right) \right\rceil_q
\]

\[
c_2 = \left\lceil \frac{t}{q} \frac{ct_0[1]ct_1[1]}{} \right\rceil_q
\]

Output \((c_0, c_1, c_2)\).
Example SEAL operations

- Homomorphic multiplication
- Relinearization
- Homomorphic addition
- Multiplication with plaintext
Why is invariant noise better than inherent noise?

<table>
<thead>
<tr>
<th></th>
<th>Inherent noise bound</th>
<th>Invariant noise bound</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial</strong></td>
<td>$B(1 + 2\delta)$</td>
<td>$\frac{t}{q}</td>
</tr>
<tr>
<td><strong>Addition</strong></td>
<td>$</td>
<td></td>
</tr>
<tr>
<td><strong>Multiplication</strong></td>
<td>$+ \frac{\delta^{j+1} - 1}{2(\delta - 1)}$</td>
<td>$+ \left(\delta t + \frac{\delta t^2}{2(\delta - 1)}\right)</td>
</tr>
<tr>
<td></td>
<td>$+ \left(\delta t + \frac{\delta t^2}{2(\delta - 1)}\right)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ \frac{3\delta t}{q}</td>
<td></td>
</tr>
<tr>
<td><strong>Rereinarization</strong></td>
<td>$</td>
<td></td>
</tr>
<tr>
<td><strong>Multiply plain</strong></td>
<td>$\frac{\delta t}{2}</td>
<td></td>
</tr>
<tr>
<td><strong>Add plain</strong></td>
<td>$</td>
<td></td>
</tr>
<tr>
<td><strong>Negation</strong></td>
<td>$</td>
<td></td>
</tr>
<tr>
<td><strong>Subtraction</strong></td>
<td>$</td>
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Encoding

In SEAL, plaintext space is $R_t = \mathbb{Z}_t[x]/(x^n + 1)$
SEAL automatic parameter selection

Input descriptions of:

- computation
- plaintext
SEAL automatic parameter selection

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The tool simulates noise growth and plaintext coefficient growth to find optimal parameters:
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- Choose $t$ as the smallest power of 2 such that decoding succeeds
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Input descriptions of:

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The tool simulates noise growth and plaintext coefficient growth to find optimal parameters:

- Sets error distribution as default
- Choose $t$ as the smallest power of 2 such that decoding succeeds
- Choose $n$ and $q$ from the default pairs as the smallest such that decryption succeeds
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Parameter selection for performance in SEAL

Choosing $n$ (and $\sigma$)

- We are essentially done
- Power of two $n$ turns out to be good for performance

Choosing $t$

- If $t$ is such that $t|(q - 1)$ then $r_t(q) = 1$
Parameter selection for performance in SEAL

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Choosing $q$

- Of the form $2^A - B$, where $B$ is a small integer
- Of the form $2n|(q - 1)$
  - In particular $4q \leq \beta$, where $\beta = 2^{64\lceil\log(q)/64\rceil}$
Homomorphic operations in SEAL

**Relinearization** ($ct = (c_0, c_1, c_2)$): Express $c_2$ in base $w$ as

$$c_2 = \sum_{i=0}^{\ell} c_2^{(i)} w^i .$$

Set

$$c_0' = c_0 + \sum_{i=0}^{\ell} \text{evk}[i][0] c_2^{(i)} ,$$

$$c_1' = c_1 + \sum_{i=0}^{\ell} \text{evk}[i][1] c_2^{(i)} ,$$

and output $(c_0', c_1')$. 
Parameter selection for performance in SEAL

Relinearization: choosing $w$ and $\ell$

- Choice only affects relinearization and evaluation key generation
- Both relinearization and lack of relinearization can introduce noise
- Typical choice is $\log w = \frac{1}{2} \log q$
- Smaller $\log w$ is worse for performance
- Automatic parameter selection allows up to $\log w = \frac{1}{10} \log q$
- Essentially open problem to determine when to relinearize
Improved performance through new variant of FV [CLPX17]

- Plaintext modulus is $x - b$ rather than $t$ [HS00]
- Plaintext space is $\mathbb{Z}/(b^n + 1)\mathbb{Z}$
- Easy encoding for integers and rationals
- Performs favourably compared to FV
The new scheme

**Encoding** \( m \)

For each \( m \in \mathcal{M} \) denote by \( \hat{m} \) a shortest polynomial with \( \|\hat{m}\| \leq (b + 1)/2 \), such that \( \hat{m}(b) = m \) modulo \( b^n + 1 \)

*Encrypt((\( p_0, p_1 \), \( m \))):* Sample \( u \leftarrow \{-1, 0, 1\} \), and \( e_0, e_1 \leftarrow \chi \). Output

\[
(c_0, c_1) = (\Delta_b \hat{m} + p_0 u + e_0, p_1 u + e_1)
\]
The new scheme

Encoding \( m \)

For each \( m \in \mathcal{M} \) denote by \( \hat{m} \) a shortest polynomial with \( \| \hat{m} \| \leq (b + 1)/2 \), such that \( \hat{m}(b) = m \) modulo \( b^n + 1 \)

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Output
\[
(c_0, c_1) = (\Delta b \hat{m} + p_0 u + e_0, p_1 u + e_1).
\]

Decrypt((\( c_0, c_1 \), \( s \))): Compute
\[
\hat{M} = \left\lfloor \frac{x - b}{q} [c_0 + c_1 s]_q \right\rfloor.
\]
Output \( m' = \hat{M}(b) \in \mathcal{M} \).
Comparison to FV

- Compare evaluation of regular circuit as in [CSVW16]
  - Do $A$ additions and one multiplication, iterated $D$ times
  - Inputs are integers of norm at most $L$
Comparison to FV

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- Security for FV and new variant is the same so we can fix $(n, q, \sigma)$
Comparison to FV

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- Goal: Find $(t, D)$ and $(b, D)$ so $D$ is maximised

- Security for FV and new variant is the same so we can fix $(n, q, \sigma)$
- Noise and plaintext growth estimates give constraints
Encoders in FV

- Family parameterised by base $B$ [DGBL+15] or Non-Adjacent Form
  - Small $B$ enables smaller $t$
  - Large $B$ enables shorter encodings
Encoders in FV

- Family parameterised by base $B$ [DGBL+15] or Non-Adjacent Form
  - Small $B$ enables smaller $t$
  - Large $B$ enables shorter encodings

- Choose NAF since it outperforms $B = 2$ and $B = 3$ [CJLL17]
Results

\[
\begin{align*}
A = 0 & \quad \max D \quad \log_2(n) \\
A = 3 & \quad \max D \quad \log_2(n) \\
A = 10 & \quad \max D \quad \log_2(n)
\end{align*}
\]
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More details

CLPX17 Hao Chen, Kim Laine, Rachel Player and Yuhou Xia. High-Precision Arithmetic in Homomorphic Encryption. ia.cr/2017/809


Thank you! / Questions?


BBGS17  N. Bindel, J. Buchmann, F. Göpfert and M. Schmidt. Estimation of the hardness of the Learning with Errors problem with a restricted number of samples. Eprint 2017/140