Parameter Selection in Ring-LWE-based Fully Homomorphic Encryption

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based on joint works with

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London-ish Lattice Coding & Crypto Meeting — September 29, 2017

Table of Contents

Setting the scene

Lattice-based crypto:

- Candidate for post-quantum crypto
- Parameter selection is a drawback

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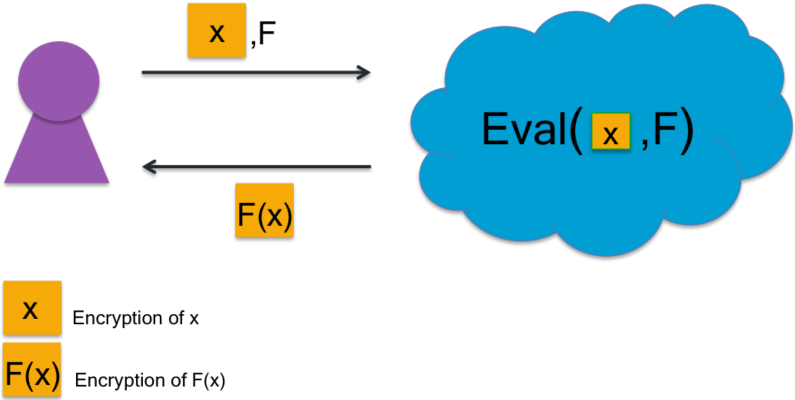
- Candidate for post-quantum crypto
- Parameter selection is a drawback

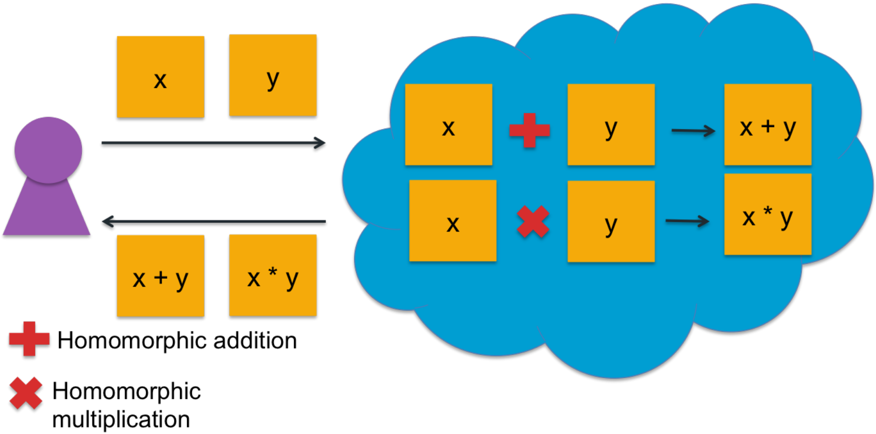
Fully Homomorphic Encryption:

- the coolest application of lattice-based crypto
- an interesting setting for parameter selection

What is homomorphic encryption?



Achieving homomorphic encryption



Applications of homomorphic encryption

- Healthcare
- Genomics
- Private set intersection
- Signal processing
- Machine learning
- ...

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  - SEAL
  - FV-NFLlib, Palisade, HEAAN, cuHE, TFHE, ...

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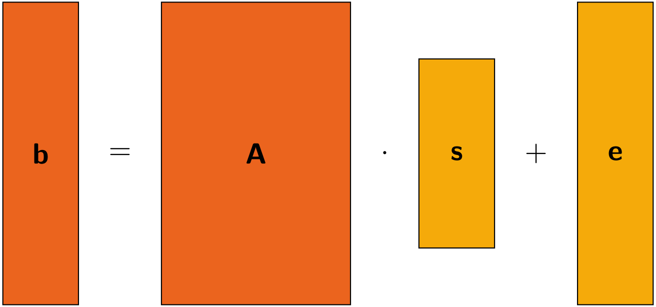
- Standardisation effort: `https://homomorphicencryption.org`
- Results for specific applications

Learning with Errors (LWE) [R05]



- Search: given $\mathbf{A}$ and $\mathbf{b}$, recover $\mathbf{s}$
- Decision: distinguish whether $(\mathbf{A}, \mathbf{b})$ is chosen as LWE or uniformly at random

Ring LWE definition

The ring R_q

Let n be a power of 2 and define

$$R_q = \mathbb{Z}_q[x]/(x^n + 1)$$

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Ring LWE (Decision)

Let $s \in R_q$ be a secret. Let $a \leftarrow R_q$ be chosen uniformly at random. Let χ be a distribution over R_q. Let $e \leftarrow \chi$. Distinguish $(a, b = as + e) \in R_q \times R_q$ from uniformly random $(a, b) \in R_q \times R_q$.

Why is n a power of two?

Theorem [LPR12]

There is a polynomial time quantum reduction from approximate SIVP (Shortest Independent Vector Problem) on ideal lattices in K to Decision Ring-LWE in R given a fixed number of samples, where the error distribution is a fixed spherical Gaussian over the field tensor product $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$.

If $n = 2^k$:

- easy to implement
- performance benefit

What are the parameters?

A (Ring) LWE instance is specified by:

- n dimension
- q modulus
- α error distribution

where the standard deviation σ of χ satisfies

$$\sigma = \frac{\alpha q}{\sqrt{2\pi}}$$

Table of Contents

Is my Ring-LWE-based scheme secure?

- Parameters n, q, α in the scheme imply an underlying Ring LWE instance

- Treat Ring LWE instance as an LWE instance

- Observe that LWE instance is hard to solve

LWE based FHE parameters are atypical

Typical LWE parameters (Regev)

- q polynomial in n
- $\alpha q = \sqrt{n}$

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FHE parameters

- huge q
- tiny error distribution e.g. $\alpha q = 8$
- small secret $\|s\| = 1$
- possibly sparse secret

So how hard is (small secret) LWE, anyway?

Theory

LWE with binary secret in dimension $n \log q$ is as hard as general LWE in dimension n. [BLP+13,MP13]

- Many approaches for solving LWE
- Even more in the case of small and/or sparse secret

[APS15] estimator for hardness of LWE instances

https://bitbucket.org/malb/lwe-estimator

input LWE instance n, q, α

output estimates of runtime, memory, samples

Can optionally specify:

- Limited samples [BBGS17]
- Secret distribution
- Lattice reduction cost method

Running example: SEAL [DGBL+15,LP16,CLP16,CLP17]

- Homomorphic encryption library
- Developed by Microsoft Research
- Current version v2.2, June 2017
- Implements FV scheme [FV12]

`sealcrypto.org`

FV is IND-CPA secure if Ring LWE is hard

SecretKeyGen: Output $s \xleftarrow{\$} R_2$

PublicKeyGen: Sample $a \xleftarrow{\$} R_q$, and $e \leftarrow \chi$. Output

$$(p_0, p_1) = ([-(as + e)]_q, a)$$

Encrypt($(p_0, p_1), m$): Sample $u \xleftarrow{\$} R_2$, and $e_1, e_2 \leftarrow \chi$. Output

$$(c_0, c_1) = ([\Delta m + p_0 u + e_1]_q, [p_1 u + e_2]_q)$$

Choosing SEAL parameters for security

Already fixed are

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Find an acceptable bit length of q

- Choose initial bit length K
- Use [APS15] estimator to determine best attack for n, $q = 2^K$, $\alpha = 8/q$
- If best attack costs less than λ, decrement K and repeat
- If best attack costs more than λ, stop

Estimate of SEAL v2.2 security [CLP17]

n	q	α	usvp	dec	dual
2048	$2^{60} - 2^{14} + 1$	$8/q$	115.5	127.1	118.4
4096	$2^{116} - 2^{18} + 1$	$8/q$	119.7	125.3	121.2
8192	$2^{226} - 2^{26} + 1$	$8/q$	123.6	126.3	124.0
16384	$2^{435} - 2^{33} + 1$	$8/q$	129.5	130.7	130.2
32768	$2^{889} - 2^{54} - 2^{53} - 2^{52} + 1$	$8/q$	127.3	127.6	127.4

Table: Estimates of the cost of solving LWE instances underlying SEAL v2.2 default parameters. Obtained using commit cc5f6e8 of the estimator in [APS15].

Table of Contents

Noise and correctness

- FHE ciphertexts all have noise
- Noise grows with homomorphic operations
- If noise too large, decryption will fail

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The better our understanding of noise the easier it is to choose parameters, and we may be able to choose smaller parameters

The FV scheme [FV12]

SecretKeyGen: Output $s \xleftarrow{\$} R_2$

PublicKeyGen(s): Sample $a \xleftarrow{\$} R_q$, and $e \leftarrow \chi$. Output

$$(p_0, p_1) = ([-(as + e)]_q, a)$$

Encrypt($(p_0, p_1), m$): Sample $u \xleftarrow{\$} R_2$, and $e_1, e_2 \leftarrow \chi$. Output

$$(c_0, c_1) = ([\Delta m + p_0 u + e_1]_q, [p_1 u + e_2]_q)$$

Decrypt(s, (c_0, c_1)): Output

$$\left[\left\lfloor \frac{t}{q}[c_0 + c_1 s]_q \right\rceil\right]_t$$

Existing notions of noise in FV

Inherent Noise [FV12,CLP16]

- The inherent noise is v_{inh} such that $[c_0 + c_1 s]_q = \Delta m + v_{inh}$.
- We require $\|v_{inh}\|_\infty < \frac{q}{2t} - \frac{t}{2}$

Critical quantity [CS16]

- The critical quantity is $v_{inh} - \frac{r_t(q)}{t} m$
- We require $\|v_{inh} - \frac{r_t(q)}{t} m\|_\infty < \frac{\Delta}{2}$

We want noise to be the thing which causes decryption to fail if it is too large

Recall FV Decryption: $m \mod t = \left[\left\lfloor \frac{t}{q} [c_0 + c_1 s]_q \right\rceil \right]_t$

Invariant noise [CLP17]

$$\frac{t}{q}(c_0 + c_1 s) = m + v + at$$

The norm $\|v\|_\infty$ is the invariant noise.

Invariant noise

By definition

$$\left\lfloor \frac{t}{q}[c_0 + c_1 s]_q \right\rceil = m + \lfloor v \rceil + a' t$$

So FV decryption succeeds

$$\left[\left\lfloor \frac{t}{q}[c_0 + c_1 s]_q \right\rceil \right]_t = m \mod t \qquad \text{if } \| v \|\|_\infty < \frac{1}{2}$$

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Noise budget

Initial noise in a fresh ciphertext is very small, and even in later ciphertexts we have $2\|v\|\|_\infty < 1$ if decryption succeeds. Easier to work with the noise budget defined as $-\log_2(2\|v\|)|_\infty$.

Homomorphic operations in SEAL

Addition $(\mathtt{ct}_0, \mathtt{ct}_1)$: Output

$$(\mathtt{ct}_0[0] + \mathtt{ct}_1[0], \mathtt{ct}_0[1] + \mathtt{ct}_1[1])$$

Multiplication $(\mathtt{ct}_0, \mathtt{ct}_1)$: Compute

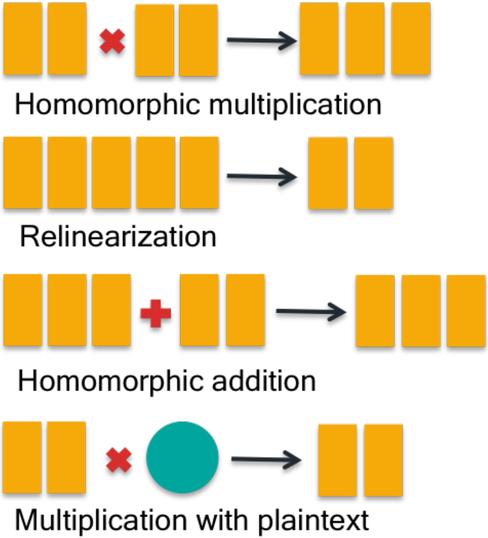
$$c_0 = \left[\left\lfloor \frac{t}{q} \mathtt{ct}_0[0]\mathtt{ct}_1[0] \right\rceil\right]_q$$

$$c_1 = \left[\left\lfloor \frac{t}{q} (\mathtt{ct}_0[0]\mathtt{ct}_1[1] + \mathtt{ct}_0[1]\mathtt{ct}_1[0]) \right\rceil\right]_q$$

$$c_2 = \left[\left\lfloor \frac{t}{q} \mathtt{ct}_0[1]\mathtt{ct}_1[1] \right\rceil\right]_q .$$

Output (c_0, c_1, c_2).

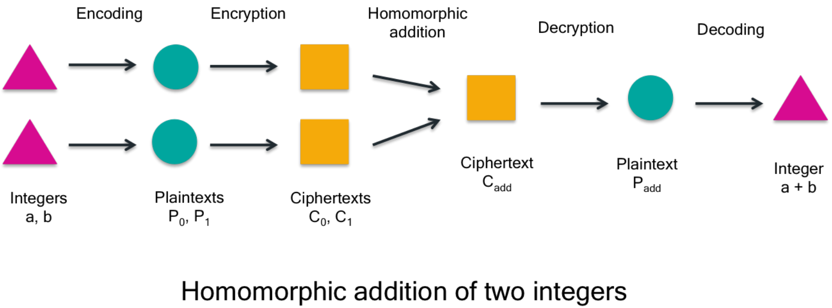
Example SEAL operations



Why is invariant noise better than inherent noise?

	Inherent noise bound	Invariant noise bound
Initial	$B(1+2\delta)$	$\frac{t}{q}\|m\| + \frac{t}{q}B(1+2\delta)$
Addition	$\|v_{1_{inh}}\| + \|v_{2_{inh}}\| + t$	$\|v_1\| + \|v_2\|$
Multiplication	$+\frac{\delta^{J+1}-1}{2(\delta-1)}$ $+\left(\delta t + \frac{\delta t(\delta^{j_2+1}-1)}{2(\delta-1)} + \frac{\delta t^2}{2q}\right)\|v_{1_{inh}}\|$ $+\left(\delta t + \frac{\delta t(\delta^{j_1+1}-1)}{2(\delta-1)} + \frac{\delta t^2}{2q}\right)\|v_{2_{inh}}\|$ $+\left(\frac{3\delta t}{q}\right)\|v_{1_{inh}}\| \cdot \|v_{2_{inh}}\|$ $\frac{2\delta t^2+t}{2} - \frac{3\delta t^3}{4q} + \frac{t^2\delta(\delta^{j_2+1}+\delta^{j_1+1}-2)}{4(\delta-1)}$	$\frac{t(\delta^{J+1}-1)}{2q(\delta-1)}$ $+\left(\delta t + \frac{\delta t(\delta^{j_2+1}-1)}{2(\delta-1)}\right)\|v_1\|$ $+\left(\delta t + \frac{\delta t(\delta^{j_1+1}-1)}{2(\delta-1)}\right)\|v_2\|$ $+3\delta \cdot \|v_1\| \cdot \|v_2\|$
Relinearization	$\|v_{inh}\| + (M-N)(\ell+1)\delta Bw$	$\|v\| + \frac{t}{q}(M-N)(\ell+1)\delta Bw$
Multiply plain	$\frac{\delta t}{2}\|v_{inh}\| + \frac{t}{2} + \frac{\delta t^2}{4}$	$\frac{\delta t}{2}\|v\|$
Add plain	$\|v_{inh}\| + t$	$\|v\| + \frac{t^2}{2q}$
Negation	$\|v_{inh}\| + t$	$\|v\|$
Subtraction	$\|v_{1_{inh}}\| + \|v_{2_{inh}}\| + t$	$\|v_1\| + \|v_2\|$

Encoding



In SEAL, plaintext space is $R_t = \mathbb{Z}_t[x]/(x^n + 1)$

SEAL automatic parameter selection

Input descriptions of:

- computation
- plaintext

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The tool simulates noise growth and plaintext coefficient growth to find optimal parameters:

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- Choose t as the smallest power of 2 such that decoding succeeds
- Choose n and q from the default pairs as the smallest such that decryption succeeds

Table of Contents

Parameter selection for performance in SEAL

Choosing n (and σ)

- We are essentially done
- Power of two n turns out to be good for performance

Choosing t

- If t is such that $t | (q-1)$ then $r_t(q) = 1$

Parameter selection for performance in SEAL

n	q
8192	$2^{226} - 2^{26} + 1$
16384	$2^{435} - 2^{33} + 1$
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Choosing q

- Of the form $2^A - B$, where B is a small integer
- Of the form $2n|(q-1)$
  - In particular $4q \leq \beta$, where $\beta = 2^{64\lceil \log(q)/64 \rceil}$

Homomorphic operations in SEAL

Relinearization ($\mathtt{ct} = (c_0, c_1, c_2)$): Express c_2 in base w as

$$c_2 = \sum_{i=0}^{\ell} c_2^{(i)} w^i \, .$$

Set

$$c_0' = c_0 + \sum_{i=0}^{\ell} \mathtt{evk}[i][0] c_2^{(i)} \, ,$$

$$c_1' = c_1 + \sum_{i=0}^{\ell} \mathtt{evk}[i][1] c_2^{(i)} \, ,$$

and output (c_0', c_1').

Parameter selection for performance in SEAL

Relinearization: choosing w and ℓ

- Choice only affects relinearization and evaluation key generation
- Both relinearization and lack of relinearization can introduce noise
- Typical choice is $\log w = \frac{1}{2} \log q$
- Smaller $\log w$ is worse for performance
- Automatic parameter selection allows up to $\log w = \frac{1}{10} \log q$
- Essentially open problem to determine when to relinearize

Improved performance through new variant of FV [CLPX17]

- Plaintext modulus is $x - b$ rather than t [HS00]
- Plaintext space is $\mathbb{Z}/(b^n + 1)\mathbb{Z}$
- Easy encoding for integers and rationals
- Performs favourably compared to FV

The new scheme

Encoding m

For each $m \in \mathcal{M}$ denote by $\widehat{m}$ a shortest polynomial with $\|\widehat{m}\| \leq (b+1)/2$, such that $\widehat{m}(b) = m$ modulo $b^n + 1$

Encrypt($(p_0, p_1), m$): Sample $u \leftarrow \{-1, 0, 1\}$, and $e_0, e_1 \leftarrow \chi$. Output

$$(c_0, c_1) = \left(\Delta_b \widehat{m} + p_0 u + e_0, p_1 u + e_1\right) .$$

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$$(c_0, c_1) = (\Delta_b \widehat{m} + p_0 u + e_0, p_1 u + e_1) .$$

Decrypt$((c_0, c_1), s)$: Compute

$$\widehat{M} = \left\lfloor \frac{x-b}{q} [c_0 + c_1 s]_q \right\rceil .$$

Output $m' = \widehat{M}(b) \in \mathcal{M}$.

Comparison to FV

- Compare evaluation of regular circuit as in [CSVW16]
  - Do A additions and one multiplication, iterated D times
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- Security for FV and new variant is the same so we can fix (n, q, σ)
- Noise and plaintext growth estimates give constraints

Encoders in FV

- Family parameterised by base B [DGBL+15] or Non-Adjacent Form
  - Small B enables smaller t
  - Large B enables shorter encodings

Encoders in FV

- Family parameterised by base B [DGBL+15] or Non-Adjacent Form
  - Small B enables smaller t
  - Large B enables shorter encodings

- Choose NAF since it outperforms $B = 2$ and $B = 3$ [CJLL17]

Results

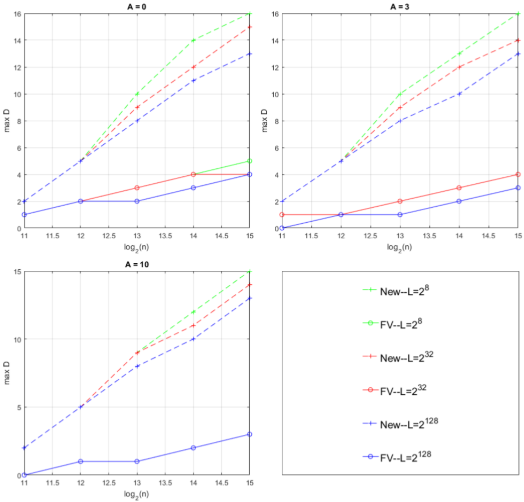
A = 0
A = 3
A = 10
max D
log_2(n)
New--L=2^8
FV--L=2^8
New--L=2^32
FV--L=2^32
New--L=2^128
FV--L=2^128

Table of Contents

More details

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Thank you! / Questions?

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