Recent Advances in Decoding Random Binary Linear Codes – and Their Implications to Crypto

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Linear Codes and Distance

Definition Linear Code

A linear code is a k-dimensional subspace of $\mathbb{F}_2^n$.

Represent via:

- **Generator matrix** G

$$C = \{\mathbf{x}G \in \mathbb{F}_2^n \mid \mathbf{x} \in \mathbb{F}_2^k\}, \text{ where } G \in \mathbb{F}_2^{k \times n}$$

- **Parity check matrix** H

$$C = \{\mathbf{c} \in \mathbb{F}_2^n \mid H\mathbf{c} = \mathbf{0}\}, \text{ where } H \in \mathbb{F}_2^{n-k \times n}$$

- **Random Code:** $G \in_R \mathbb{F}_2^{k \times n}$ respectively $H \in_R \mathbb{F}_2^{n-k \times n}$
    - Random codes are hard instances for decoding.
    - Crypto motivation: Scramble structured C in “random" SCT.
    - Good generic hardness criterion.

Bounded and Full Distance Decoding

Definition Distance

$d = \min_{\mathbf{c} \neq \mathbf{c}' \in C}\{\Delta(\mathbf{c}, \mathbf{c}')\}$, where Δ is the Hamming distance.

Remark: Unique decoding of $\mathbf{c} + \mathbf{e}$ when $\Delta(\mathbf{e}) \leq \frac{d-1}{2}$.

Definition Bounded Distance Decoding (BD)

Given : $H, \mathbf{x} = \mathbf{c} + \mathbf{e}$ with $\mathbf{c} \in C, \Delta(\mathbf{e}) \leq \frac{d-1}{2}$
Find : $\mathbf{e}$ and thus $\mathbf{c} = \mathbf{x} + \mathbf{e}$

Syndrome Decoding

- Syndrome $\mathbf{s} := H\mathbf{x} = H(\mathbf{c} + \mathbf{e}) = H\mathbf{c} + H\mathbf{e} = H\mathbf{e}$.
- Bounded Distance is the usual case in crypto.

Definition Full Distance Decoding (FD)

Given : $H, \mathbf{x} \in \mathbb{F}_2^n$
Find : $\mathbf{c}$ with $\Delta(\mathbf{c}, \mathbf{x}) \leq d$

On Running Times

- Running time of any decoding algorithm is a function of (n, k, d).
- Look at map $\mathbb{F}_2^n \to \mathbb{F}_2^{n-k}$ with $\mathbf{e} \mapsto H\mathbf{e}$ with $\Delta(\mathbf{e}) \leq d$.
- Map is injective if $\binom{n}{d} < 2^{n-k}$.
- Write $\binom{n}{d} \approx 2^{H(\frac{d}{n})n}$, which yields

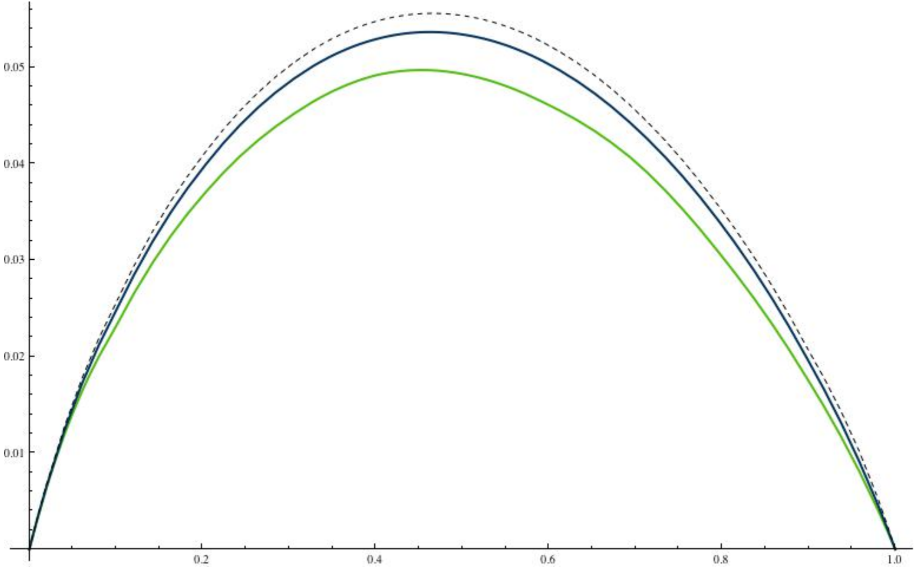
$$H(\tfrac{d}{n}) < 1 - \tfrac{k}{n}. \quad \text{(Gilbert-Varshamov bound)}$$

- For **random codes** this bound is sharp.
- Hence, we can directly link d to n, k.
- Running time becomes a function of n, k only.
- Since BD/FD decoding is NP-hard we expect running time

$$T(n, k) = 2^{f(\frac{k}{n})n}.$$

- For simplifying, we are mainly interested in $T(n) = \max_k\{T(n, k)\}$.

Running Time graphically



The Way to go

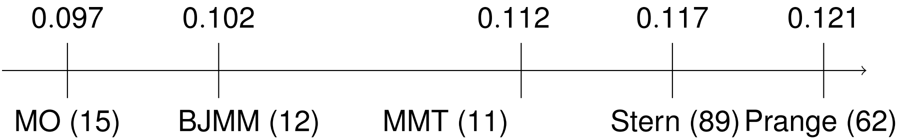


Figure: Full Distance decoding (FD)

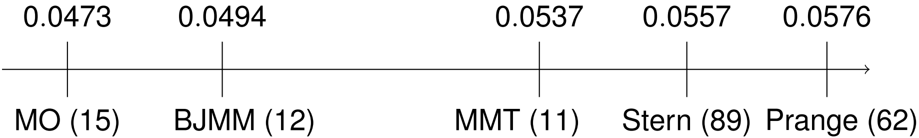


Figure: Bounded Distance decoding (BD)

Let's just start.

Goal: Solve $H\mathbf{e} = \mathbf{s}$ for small weight $\mathbf{e}$.
Assumption: Wlog we know $\omega := \Delta(\mathbf{e})$.

Algorithm Exhaustive Search

INPUT: H, $\mathbf{x}$, ω

1. For all $\mathbf{e} \in \mathbb{F}_2^n$ with $\Delta(\mathbf{e}) = \omega$: Check whether $H\mathbf{e} = \mathbf{s} = H\mathbf{x}$.

OUTPUT: e

Running time: $T(n) = \binom{n}{\omega} \leq 2^{0.386n}$.

Allowed Transformations

Linear algebra transformation for $\mathbf{s} = H\mathbf{e}$.

1. **Column permutation**:
$$\mathbf{s} = H\mathbf{e} = HPP^{-1}\mathbf{e}$$
for some permutation matrix $P \in \mathbb{F}_2^{n \times n}$.
2. **Elementary row operations**:
$$GH\mathbf{e} = G\mathbf{s} =: \mathbf{s}'$$
for some invertible matrix $G \in \mathbb{F}_2^{n-k \times n-k}$.

Easy special cases:

1. **Quadratic case**: $H \in \mathbb{F}_2^{n \times n}$. Compute $\mathbf{e} = H^{-1}\mathbf{s}$.
2. **Any weight** $\Delta(\mathbf{e})$: Compute $GH\mathbf{e} = (H' \mid I_{n-k})\mathbf{e} = G\mathbf{s}$.

Remark: Hardness/unicity comes from under-defined + small weight.

Prange's algorithm (1962)

Idea: $(H' \mid I_{n-k})(\mathbf{e}_1 \mid\mid \mathbf{e}_2) = H'\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{s}'$

Algorithm Prange

INPUT: H, $\mathbf{x}$, ω

REPEAT

1. Permute columns, construct systematic $(H' \mid I_{n-k})$. Fix $p < \omega$.
2. For all $\mathbf{e}_1 \in \mathbb{F}_2^k$ with $\Delta(\mathbf{e}_1) = p$:
    1. If $(\Delta(H'\mathbf{e}_1 + \mathbf{s}') = \omega - p)$, success.

UNTIL success

OUTPUT: Undo permutation of $\mathbf{e} = (\mathbf{e}_1 || H'\mathbf{e}_1 + \mathbf{s}')$.

Running time:

- Outer loop has success prob $\frac{\binom{k}{p}\binom{n-k}{\omega-p}}{\binom{n}{\omega}}$.
- Inner loop has running time $\binom{k}{p}$. Total: $\frac{\binom{n}{\omega}}{\binom{n-k}{\omega-p}}$, optimal for $p = 0$.
- Yields running time $T(n) = 2^{\frac{1}{17}n}$, with constant memory.

Stern's algorithm (1989)

Meet in the Middle:

$$(H_1 \mid H_2 \mid I_{n-k})(\mathbf{e}_1||\mathbf{e}_2||\mathbf{e}_3) = H_1\mathbf{e}_1 + H_2\mathbf{e}_2 + \mathbf{e}_3 = \mathbf{s}'$$

Algorithm Stern

INPUT: H, $\mathbf{x}$, ω

REPEAT

1. Permute columns, construct systematic $(H_1 \mid H_2 \mid I_{n-k})$. Fix $p < \omega$.
2. For all $\mathbf{e}_1 \in \mathbb{F}_2^{\frac{k}{2}}$ with $\Delta(\mathbf{e}_1) = \frac{p}{2}$: Store $H_1\mathbf{e}_1$ in sorted L_1.
3. For all $\mathbf{e}_2 \in \mathbb{F}_2^{\frac{k}{2}}$ with $\Delta(\mathbf{e}_2) = \frac{p}{2}$: Store $H_2\mathbf{e}_2 + \mathbf{s}'$ in sorted L_2.
4. Search for elements in L_1, L_2 that differ by $\Delta(\mathbf{e}_3) = \omega - p$.

UNTIL success

OUTPUT: Undo permutation of $\mathbf{e} = (\mathbf{e}_1||\mathbf{e}_2||H_1\mathbf{e}_1 + H_2\mathbf{e}_2 + \mathbf{s}')$.

- Step 4: Look for vectors that completely match in ℓ coordinates.
- $T(n) = 2^{\frac{1}{18}}$, but requires memory to store L_1, L_2.

Representation Technique (Howgrave-Graham, Joux)

Meet in the Middle

- Split $\mathbf{e} = (\mathbf{e}_1 || \mathbf{e}_2)$ as $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{F}_2^{\frac{k}{2}}$ with weight $\Delta(\mathbf{e}_i) = \frac{p}{2}$ each.
- Combination of $\mathbf{e}_1, \mathbf{e}_2$ is via concenation.
- Unique representation of $\mathbf{e}$ in terms of $\mathbf{e}_1, \mathbf{e}_2$.

Representation [May, Meurer, Thomae 2011]

- Split $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ as $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{F}_2^k$ with weight $\Delta(\mathbf{e}_i) = \frac{p}{2}$ each.
- Combination of $\mathbf{e}_1, \mathbf{e}_2$ is via addition in $\mathbb{F}_2^k$.
- $\mathbf{e}$ has many representations as $\mathbf{e}_1 + \mathbf{e}_2$.

Example for $k = 8, p = 4$:

$$\begin{aligned}(01101001) &= (01100000) + (00001001)\\ &= (01001000) + (00100001)\\ &= (01000001) + (00101000)\\ &= (00101000) + (01000001)\\ &= (00100001) + (01001000)\\ &= (00001001) + (01100000)\end{aligned}$$

Pros and Cons of representations

Representation [MMT 2011, Asiacrypt 2011]

- Split $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ as $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{F}_2^k$ with weight $\Delta(\mathbf{e}_i) = \frac{p}{2}$ each.
- **Disadvantages:**
  - List lengths of L_1, L_2 increases from $\binom{k/2}{p/2}$ to $\binom{k}{p/2}$.
  - Addition of $\mathbf{e}_1, \mathbf{e}_2$ usually yields Hamming weight smaller p.
- **Advantage:**
  - $\mathbf{e}$ has $\binom{p}{p/2} =: R$ representations as $\mathbf{e}_1 + \mathbf{e}_2$.
- Construct via Divide & Conquer only $\frac{1}{R}$-fraction of L_1, L_2.
- Since many solutions exist, it is easier to construct a special one.
- **Example:** Look only for $H_1\mathbf{e}_1, H_2\mathbf{e}_2 + \mathbf{s}'$ with last $\log(\frac{1}{R})$ coord. 0.
- Advantage (may) dominate whenever

$$\frac{\binom{k}{p/2}}{\binom{p}{p/2}} < \binom{k/2}{p/2}.$$

Result: Yields running time $2^{\frac{1}{19}n}$.

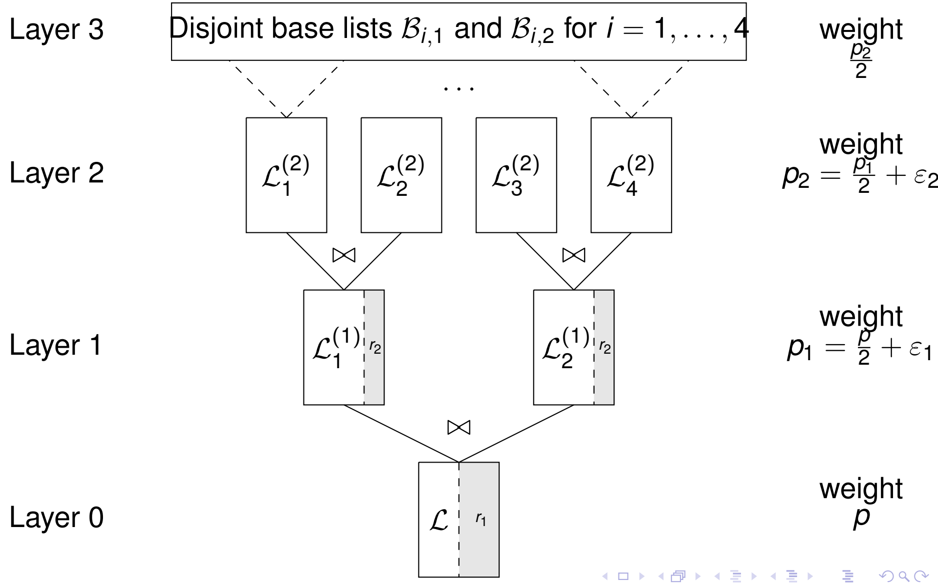
More representations (Becker,Joux,May,Meurer 2012)

Idea:

- Choose $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{F}_2^k$ with weight $\Delta(\mathbf{e}_i) = \frac{p}{2} + \epsilon$ each.
- Choose ϵ such that ϵ 1-positions cancel on expectation.
- In MMT: $\binom{p}{p/2}$ representations of 1's as
$$1 = 1 + 0 = 0 + 1.$$
- Now: Additionally $\binom{k-p}{\epsilon}$ representations of 0's as
$$0 = 1 + 1 = 0 + 0.$$
- Paper subtitle:

  "How $1 + 1 = 0$ Improves Information Set Decoding".
- Yields $T(n) = 2^{\frac{1}{20}n}$.

How to construct special solutions



A word about memory

	Bounded Distance		Full Distance	
	time	space	time	space
Prange	0.05752	-	0.1208	-
Stern	0.05564	0.0135	0.1167	0.0318
Ball-collision	0.05559	0.0148	0.1164	0.0374
MMT	0.05364	0.0216	0.1116	0.0541
BJMM	0.04934	0.0286	0.1019	0.0769

Stern's algorithm (1989)

Meet in the Middle:

$$(H_1 \mid H_2 \mid I_{n-k})(\mathbf{e}_1 || \mathbf{e}_2 || \mathbf{e}_3) = H_1\mathbf{e}_1 + H_2\mathbf{e}_2 + \mathbf{e}_3 = \mathbf{s}'$$

Algorithm Stern

INPUT: H, $\mathbf{x}$, ω

REPEAT

1. Permute columns, construct systematic $(H_1 \mid H_2 \mid I_{n-k})$. Fix $p < \omega$.
2. For all $\mathbf{e}_1 \in \mathbb{F}_2^{\frac{k}{2}}$ with $\Delta(\mathbf{e}_1) = \frac{p}{2}$: Store $H_1\mathbf{e}_1$ in sorted L_1.
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4. Search for elements in L_1, L_2 that differ by $\Delta(\mathbf{e}_3) = \omega - p$.

UNTIL success

OUTPUT: Undo permutation of $\mathbf{e} = (\mathbf{e}_1 || \mathbf{e}_2 || H_1\mathbf{e}_1 + H_2\mathbf{e}_2 + \mathbf{s}')$.

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Nearest Neighbor Problem

Definition Nearest Neighbor Problem

Given : $L_1, L_2 \subset_R \mathbb{F}_2^n$ with $|L_i| = 2^{\lambda n}$
Find : all $(\mathbf{u}, \mathbf{v}) \in L_1 \times L_2$ with $\Delta(\mathbf{u}, \mathbf{v}) = \gamma n$.

Easy cases:

1. $\gamma = \frac{1}{2}$
   - Test every combination in $L_1 \times L_2$.
   - Run time $2^{2\lambda n(1+o(1))}$.
2. $\gamma = 0$
   - Sort lists and find matching pairs.
   - Run time $2^{\lambda n(1+o(1))}$.

Theorem May, Ozerov 2015

Nearest Neighbor can be solved in $2^{\frac{1}{1-\gamma}\lambda n(1+o(1))}$.

Main Idea of Nearest Neighbor

Observation: Nearest Neighbors are also **locally** near.

$\mathbf{u} \in \mathbf{L}_1$
$\mathbf{v} \in \mathbf{L}_2$

$\boxed{\mathbf{L}_1}\boxed{\mathbf{L}_2}$ size: $2^{\lambda n}$

create exponentially many sublists
by choosing random partitions P

$\boxed{\mathbf{L}_1'}\boxed{\mathbf{L}_2'}$ $\boxed{\mathbf{L}_1'}\boxed{\mathbf{L}_2'}$ $\cdots$ $\boxed{\mathbf{L}_1'}\boxed{\mathbf{L}_2'}$ $\boxed{\mathbf{L}_1'}\boxed{\mathbf{L}_2'}$

For at least one sublist pair we have $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}_1' \times \mathbf{L}_2'$ w.o.p.

Nearest Neighbor algorithm

Algorithm Nearest Neighbor

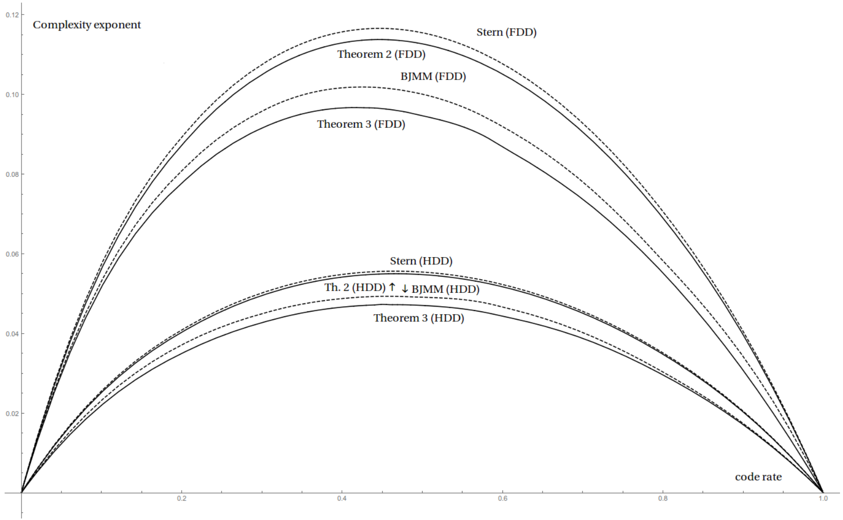
INPUT: $L_1, L_2 \subset_R \mathbb{F}_2^n$
REPEAT sufficiently often:

1. Randomly compute a partition P of $[n]$.
2. For each set $p \in P$
    1. Compute weight in a random half of the p-coordinates of L_1, L_2.
    2. Keep only those vectors with a certain weight (depending on γ).
3. Search the remaining filtered lists naively.

OUTPUT: all $(\mathbf{u}, \mathbf{v}) \in L_1 \times L_2$ with $\Delta(\mathbf{u}, \mathbf{v}) = \gamma n$

- Filters out until L_1, L_2 reach polynomial size.
- Algorithm has quite large polynomial overheads.
- Yields $T(n) < 2^{\frac{1}{21}n}$ for Bounded Distance Decoding.

Improvements graphically



Asymptotical or Real?

Yann Hamdaoui and Nicolas Sendrier,
"A Non Asymptotic Analysis of Information Set Decoding", 2013

(n, k, d)	Stern	MMT	BJMM
(1024, 524, 50)	55.60	54.75	52.90
(2048, 1696, 32)	81.60	79.50	76.82
(4096, 3844, 21)	81.23	78.88	78.46

Asymptotics for Defended McEliece

(n, k, ω)	Security	w/o NN	w/ NN
(1632, 1269, 34)	80	59	57
(2960, 288, 57)	128	107	104
(4096, 3844, 117)	256	240	232

Conclusion

MMT, BJMM relevant for cryptographic keysizes! Breakpoint for MO?

But: The improvements asymptotically vanish for McEliece.

The LPN Problem and its Relation to Codes

Problem Learning Parities with Noise ($\text{LPN}_{n,p}$)

Given: $(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) \in \mathbb{F}_2^n \times \mathbb{F}_2$ with $\Pr[e_i = 1] = p$.
Find: $\mathbf{s} \in \mathbb{F}_2^n$

- Notation: $A\mathbf{s} = \mathbf{b} + \mathbf{e}$. For $p = 0$: Compute $\mathbf{s} = A^{-1}\mathbf{b}$.
- Best algorithm: BKW with time/sample/space $2^{\frac{n}{\log(\frac{n}{p})}}$.

Algorithm GAUSS

1. REPEAT
   1. Take n fresh samples. Compute $\mathbf{s}' = A^{-1}\mathbf{b}$.
2. UNTIL $\mathbf{s}' = \mathbf{s}$

Theorem

GAUSS runs in time/sample complexity $\left(\frac{1}{1-p}\right)^n$ and poly space.

Proof: $\Pr[\text{Iteration of REPEAT successful}] = (1-p)^n$.

Getting the samples down.

Algorithm POOLED GAUSS (Esser, Kübler, May – Crypto 2017)

1. Choose a pool of $\Theta(n^2)$ samples.
2. REPEAT
   1. Take n samples from the pool. Compute $\mathbf{s}' = A^{-1}\mathbf{b}$.
3. UNTIL $\mathbf{s}' = \mathbf{s}$

Theorem

POOLED GAUSS runs in time $\left(\frac{1}{1-p}\right)^n$ with poly samples/space.

Theorem

POOLED GAUSS **quantumly** runs in $\left(\frac{1}{1-p}\right)^{\frac{n}{2}}$ with poly samples/space.

Corollary

Let $p(n) \to 0$. Then POOLED GAUSS runs in e^{pn}.

Decoding LPN with Preprocessing

Algorithm LPN with Preprocessing

INPUT: $\text{LPN}_{n,p}$ instance

1. **Modify**: Use many samples to produce pool of dim-reduced ones. Results in $\text{LPN}_{n',p'}$ instance with $n' < n$ and $p' \geq p$, e.g. use BKW.
2. **Decode**: Use decoding to solve $\text{LPN}_{n',p'}$, e.g. POOLED GAUSS.
3. **Complete**: Recover rest of **s**, e.g. via enumeration or iterating.

Yields HYBRID algorithm that optimally uses space.

- For polynomial space: Put all efforts in **Decode**.
- For arbitrary space: Put all efforts in **Modify**.

Bit Complexity Estimates for Memory $\leq 2^{60}$

Largest RAM today: IBM 20-Petaflops with 1.6PB$< 2^{54}$ bits.

Table: HYBRID

p	n							
	256	384	448	512	576	640	768	1280
$\frac{1}{\sqrt{n}}$	**46**	**53**	**56**	**59**	**62**	**64**	**68**	**82**
0.05	**42**	**53**	**58**	**63**	**68**	**73**	**82**	**120**
0.125	60	**88**	**99**	**110**	**121**	**132**	**154**	**239**
0.25	81	**139**	**158**	**178**	**197**	**216**	**255**	**407**
0.4	108	174	207	240	**273**	**300**	**355**	**575**

Bit Complexity Estimates for Memory $\leq 2^{60}$

Table: Well-Pooled MMT

p	n							
	256	384	448	512	576	640	768	1280
$\frac{1}{\sqrt{n}}$	37	42	45	47	48	51	54	66
0.05	33	43	48	57	58	62	70	102
0.125	57	77	88	97	102	118	138	219
0.25	92	128	148	166	185	204	242	392
0.4	129	183	211	238	265	292	347	568

NIST Security Levels

Table: $p = \frac{1}{8}$

n	Classic	Quantum
715	**128**	90
1115	**192**	127
1520	**256**	164
450	86	**64**
615	112	**80**
1130	194	**128**

Table: $p = \frac{1}{4}$

n	Classic	Quantum
386	**128**	91
602	**192**	130
810	**256**	167
243	87	**64**
330	112	**80**
594	190	**128**

Experiments

Table: Solved instances

Algorithm	n	p	Pool	BKW	Decode	**Total**
WP MMT	243	0.125	6.73 d	-	8.34 d	**15.07 d**
WP MMT	135	0.25	5.65 d	-	8.19 d	**13.84 d**
Hybrid	135	0.25	2.21 d	1.72 h	3.41 d	**5.69 d**

Conclusions and Questions

- Improvement for BD
$$2^{\frac{1}{17}n} \to 2^{\frac{1}{18}n} \to 2^{\frac{1}{19}n} \to 2^{\frac{1}{20}n} \to 2^{\frac{1}{21}n}.$$
- Extensions to codes over $\mathbb{F}_q$ possible, but less effective.
- More applications of representations, nearest neighbors?
- May threaten McEliece security. Implementations?
- LPN with $n = 512, p = \frac{1}{4}$ or even $p = \frac{1}{8}$ seems (practically) secure.
- Generalization of LPN to LWE decoding only good for small error.
- Cryptanalysis: **Real implementations + extrapolation.**
- There is a need for small memory algorithms. What is small?
- Rule of thumb: If using time $T = 2^n$, limit memory to $M = 2^{\frac{n}{2}}$?

On the Shape of Cryptanalysis

Usefulness of Cryptanalysis:

- Provable security never solves problems, but transfers them.
- Eventually one has to use Cryptanalysis for finding keys!
- Cryptanalysis is useful, and will be in the future.
- Only 5-10% of papers is Cryptanalysis.

How to do Cryptanalysis:

- Do real experiments on small to medium scale!
- Extrapolate to large scale by asymptotical analysis.
- Asymptotical improvements are relevant improvements.
- Changing the constant *in the exponent* is significant!

On the Shape of Cryptanalysis

What you should **avoid** in Cryptanalysis:

- Pseudo-concrete estimates using strange counting of steps.
- Your algorithm requires only $2^{79.99}$ operations for 80-bit security.
- As a reviewer:
  - “After 30 pages of proofs, I need convincing experiments”.
  - “You did better, but do not cite my work. Reject."
- Do not outsource cryptanalysis to other fields.

Why you should work in Cryptanalysis:

- You really solve problems, and not relate them to others.
- You can implement your algorithm, let it run and output solutions.
- It is fun to destroy things!

If you absolutely hate Cryptanalysis, still **encourage** it.

- If you invent a scheme, instantiate it with parameters.