Lattice Reduction and Factorization for Equalization

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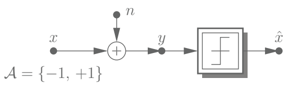
Outline

Introduction

Digital Communications:

- abstract high-level view of digital communications
  - a *point* x drawn from some *signal constellation* $\mathcal{A}$ is transmitted
    (a point can represent $\log_2 |\mathcal{A}|$ bits of information)
  - the channel adds (interference and) noise n
  - the received symbols is $y = x + n$
  - at the receiver, *decisions* have to be taken



- since we can use *quadrature modulation* (modulation of amplitude and phase), all signals are *complex-valued*

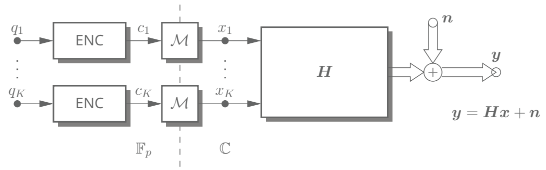
Channel Coding:

- for reducing the error rate, channel coding is employed
- in block codes (codelength η) not all $\mathcal{A}^\eta$ combinations are used but only those which can be distinguished reliably
- a trade-off between transmission rate (bit rate) and error rate is possible

Introduction (II)

Situation: multipoint-to-point transmission, *MIMO multiple-access channel*

- K non-cooperating single-antenna users
- central base station with N_R receive antennas
  ⇒ *joint processing/decoding at the receiver side possible*

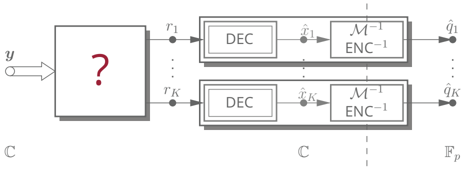


Channel Encoding / Mapping:

- channel coding done over the finite field $\mathbb{F}_p$
  (q_k and c_k taken from $\mathbb{F}_p$)
- mapping $\mathcal{M}$ of finite-field symbols c_k to complex-valued points x_k taken from some signal constellation $\mathcal{A}$

Introduction (III)

Question: *How to perform equalization / decoding?*



Usual Approach:

- joint equalization / decoding typically much to complex
  - ⇒ *separate equalization / decoding*
- channel decoding
  - individual (per user)
  - over a temporal block (code word)
- low-complexity equalization strategy (as for the uncoded case)
  - over the users
  - per time step

Introduction (IV)

Equalization of MIMO Channel:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$$

done symbol-by-symbol (independently over the time steps) in the uncoded case

Equalization Schemes:

- linear equalization
  according to zero-forcing (ZF) or minimum mean-squared error (MMSE) criterion
- decision-feedback equalization (DFE)
  aka successive interference cancellation, (V-)BLAST
- lattice-reduction-aided (LRA) / integer-forcing (IF) schemes
  low-complexity, high-performance schemes
- maximum-likelihood detection (MLD) / lattice decoding
  optimum procedure, highest complexity

Signal Constellations and Codes

Signal Constellation: Construction

- signal point lattice

$$\boldsymbol{\Lambda}_{\rm a}$$

  typically: $\boldsymbol{\Lambda}_{\rm a} = \mathbb{Z}$ or $\boldsymbol{\Lambda}_{\rm a} = \mathbb{G} = \mathbb{Z} + \mathrm{j}\,\mathbb{Z}$

- „shaping" lattice

$$\boldsymbol{\Lambda}_{\rm s}$$

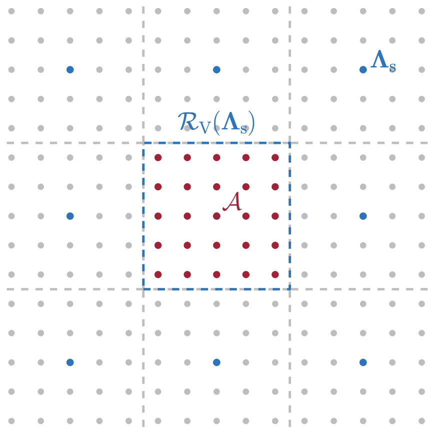
  and its Voronoi region $\mathcal{R}_{\rm V}(\boldsymbol{\Lambda}_{\rm s})$
  (typically a sublattice of $\boldsymbol{\Lambda}_{\rm a}$: $\boldsymbol{\Lambda}_{\rm s} \subset \boldsymbol{\Lambda}_{\rm a}$)

- *signal constellation*

$$\mathcal{A} = \boldsymbol{\Lambda}_{\rm a} \cap \mathcal{R}_{\rm V}(\boldsymbol{\Lambda}_{\rm s})$$

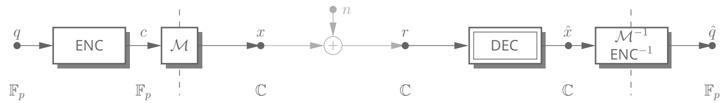
- *lattice code*
  do everything in N dimensions

$$\mathcal{C} = \boldsymbol{\Lambda}_{\rm a} \cap \mathcal{R}_{\rm V}(\boldsymbol{\Lambda}_{\rm s})$$



Decoding and Demapping

Channel Encoding and Decoding:



Enoding:

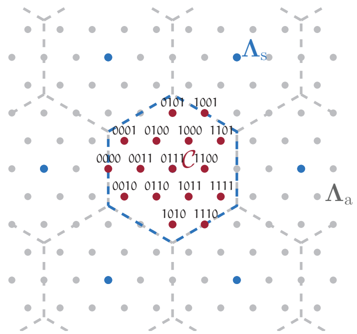
- encoding ENC over $\mathbb{F}_p$
- mapping $\mathcal{M}$ to signal point in $\mathbb{C}$

Decoding:

- lattice decoding (in signal space) w.r.t. to $\boldsymbol{\Lambda}_{\rm c}$
- demapping $\mathcal{M}^{-1}$ to $\hat{c} \in \mathbb{F}_p$
- encoder inverse ENC^{-1}

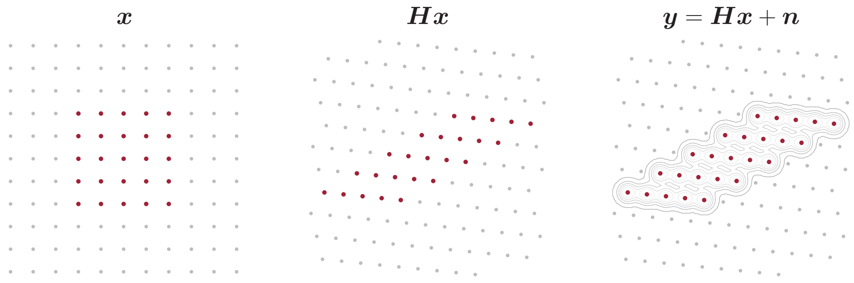
Variant:

- demapping modulo $\boldsymbol{\Lambda}_{\rm s}$, i.e., $\mathrm{mod}\mathcal{M}^{-1}$



Structure of the Signals

Visualization: (real-valued example $K = 2$, $\Lambda_c = \mathbb{Z}$, $|\mathcal{A}| = 5$)



Structure of the Signals (II)

Lattice:

- K-dim. lattice spanned by basis vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_K$ — basis matrix

$$\boldsymbol{B} = \left[\boldsymbol{b}_1 \; \boldsymbol{b}_2 \; \cdots \; \boldsymbol{b}_K\right]$$

- real-valued lattice

$$\boldsymbol{\Lambda} = \left\{\boldsymbol{\lambda} = \sum_{k=1}^{K} z_k \boldsymbol{b}_k = \boldsymbol{B}\begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix} \mid z_k \in \mathbb{Z}\right\} \stackrel{\text{def}}{=} \boldsymbol{B}\mathbb{Z}^K$$

Lattice Structure of the Signal:

- for $\boldsymbol{x} \subset \mathbb{G}^K = (\mathbb{Z} + \mathrm{j}\mathbb{Z})^K$ the noise-free receive vectors

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x}$$

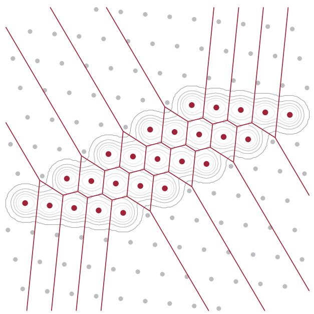
are taken from the complex-valued lattice $\boldsymbol{\Lambda} = \boldsymbol{H}\mathbb{G}^K$ spanned by the columns $\boldsymbol{h}_k$ of the channel matrix

$$\boldsymbol{H} = \left[\boldsymbol{h}_1 \; \boldsymbol{h}_2 \; \cdots \; \boldsymbol{h}_K\right]$$

Maximum-Likelihood Detection

Optimum Detection Rule: *ML criterion* $\quad \mathrm{f}_X(x)$: probability density function

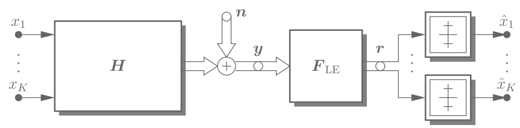
$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x} \in \mathcal{A}^K}{\operatorname{argmax}} \, \mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}) = \underset{\boldsymbol{x} \in \mathcal{A}^K}{\operatorname{argmin}} \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2$$



- *lattice decoding* — high complexity per time step
  efficient implementation via the *Sphere Decoder* [AEVZ'02]
- for combination with channel decoding generation of soft output required

Linear Equalization

Linear Equalization: simple strategy — filtering followed by individual decision/decoding



- this equalization *strategy / scheme* can be optimized either according to the *zero-forcing (ZF)* or *minimum mean-squared error (MMSE) criterion*
- zero-forcing criterion: ($\boldsymbol{I}$: identity matrix; $(\cdot)^+$: (left) pseudoinverse)

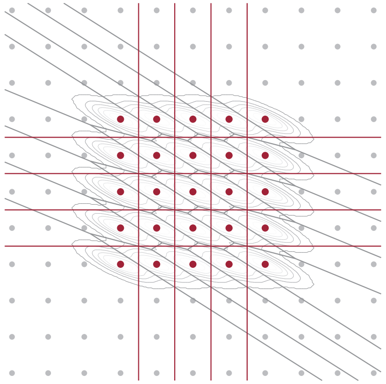
$$\boldsymbol{F}_{\mathrm{LE}} \cdot \boldsymbol{H} \stackrel{!}{=} \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{F}_{\mathrm{LE,ZF}} = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathsf{H}} \stackrel{\text{def}}{=} \boldsymbol{H}^{+}$$

- minimum mean-squared error criterion: ($\zeta \stackrel{\text{def}}{=} \sigma_n^2/\sigma_x^2$)
  error signal $\boldsymbol{e} = \boldsymbol{F}_{\mathrm{LE}}\boldsymbol{y} - \boldsymbol{x}$; error covariance matrix $\boldsymbol{\Phi}_{ee} = \mathrm{E}\{\boldsymbol{e}\boldsymbol{e}^{\mathsf{H}}\}$

$$\operatorname{trace}\left(\boldsymbol{\Phi}_{ee}\right) \stackrel{!}{\to} \min \quad \Rightarrow \quad \boldsymbol{F}_{\mathrm{LE,MMSE}} = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1}\boldsymbol{H}^{\mathsf{H}}$$

Linear Equalization (II)

Visualization:



Problem of equalizing the signal

- the noise is filtered, too $\Rightarrow$ ***noise enhancement***
- individual threshold decision per dimension not optimum

Linear Equalization (III)

Noise Enhancement:

- ZF solution — $\boldsymbol{F}_{\mathrm{LE,ZF}} = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathsf{H}} = \begin{bmatrix}\boldsymbol{f}_1\\ \vdots \\ \boldsymbol{f}_K\end{bmatrix}$; $\quad \boldsymbol{r} = \boldsymbol{F}_{\mathrm{LE,ZF}}\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{F}_{\mathrm{LE,ZF}}\boldsymbol{n}$
  - noise variance ($\boldsymbol{n}$ i.i.d. components with variance σ_n^2)

$$\sigma_{n_k}^2 = \sigma_n^2 \cdot \|\boldsymbol{f}_k\|^2$$

  - noise enhancement

$$E_k = \sigma_{n_k}^2/\sigma_n^2 = \|\boldsymbol{f}_k\|^2$$

- (biased) MMSE solution — $\boldsymbol{F}_{\mathrm{LE,MMSE}} = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1}\boldsymbol{H}^{\mathsf{H}}$

  or with $\boldsymbol{\mathcal{H}} = \begin{bmatrix}\boldsymbol{H}\\ \sqrt{\zeta}\boldsymbol{I}\end{bmatrix}$ we have $\boldsymbol{\mathcal{F}}_{\mathrm{LE,MMSE}} = (\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}})^{-1}\boldsymbol{\mathcal{H}}^{\mathsf{H}} = \begin{bmatrix}\mathfrak{f}_1\\ \vdots \\ \mathfrak{f}_K\end{bmatrix}$
  - error covariance matrix

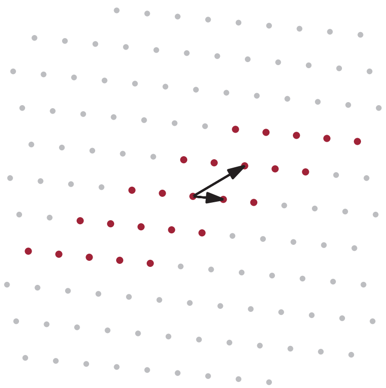
$$\boldsymbol{\Phi}_{ee}/\sigma_n^2 = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1} = \left(\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}}\right)^{-1}$$

  - noise enhancement ($\boldsymbol{\mathcal{F}}_{\mathrm{LE,MMSE}}\boldsymbol{\mathcal{F}}_{\mathrm{LE,MMSE}}^{\mathsf{H}} = (\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}})^{-1}\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}})^{-1} = (\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}})^{-1}$)

$$E_k = \left[\boldsymbol{\Phi}_{ee}/\sigma_n^2\right]_{k,k} = \|\mathfrak{f}_k\|^2$$

Lattice-Reduction-Aided Equalization

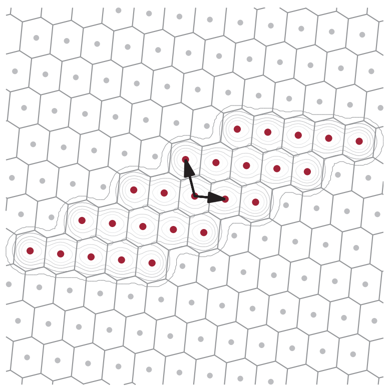
Visualization:



$$\boldsymbol{H} = \left[\boldsymbol{h}_1\ \boldsymbol{h}_2\right]$$

Lattice-Reduction-Aided Equalization

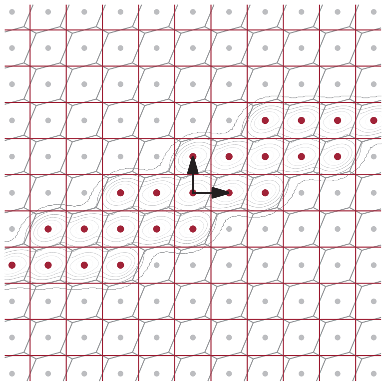
Visualization:



$$\begin{aligned}\boldsymbol{H} &= \left[\boldsymbol{h}_1\ \boldsymbol{h}_2\right]\\ \boldsymbol{C} &= \left[\boldsymbol{c}_1\ \boldsymbol{c}_2\right]\\ &= \boldsymbol{H}\boldsymbol{Z}\,, \qquad \begin{matrix}\boldsymbol{Z}\in\mathbb{Z}^{2\times 2}\\ |\det(\boldsymbol{Z})|=1\end{matrix}\end{aligned}$$

Lattice-Reduction-Aided Equalization

Visualization:



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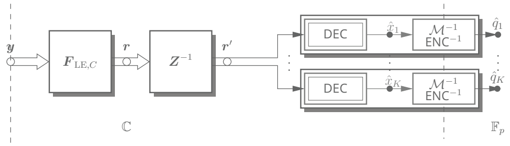
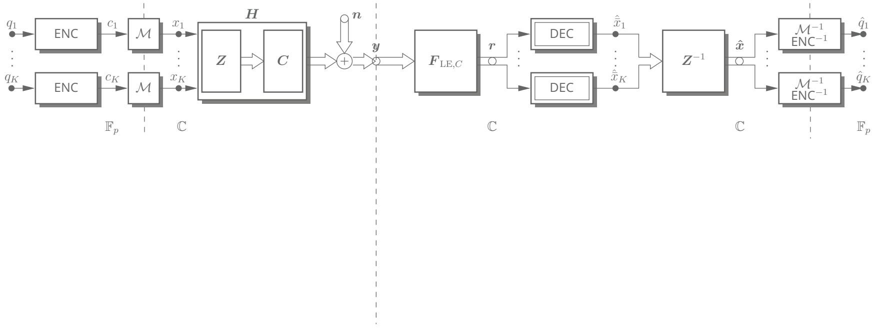
Equalization Schemes

Linear Equalization:

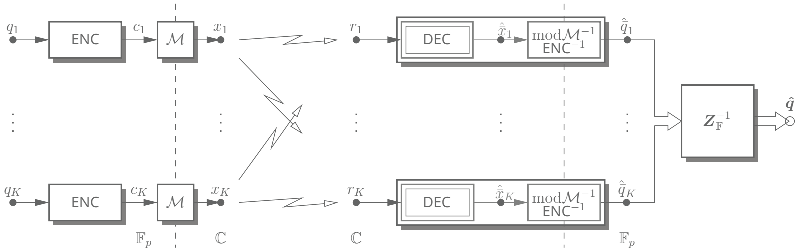


Lattice-Reduction-Aided Equalization: [YW'02], [WF'03]



Integer-Forcing Schemes

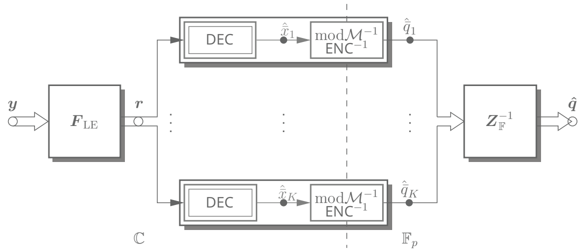
Compute-And-Forward Strategy in Relaying: [NG'11]



- the receiver decodes an *integer linear combination* of the codewords
- resolution of linear combinations at some central unit
  only finite-field symbols are communicated — processing over $\mathbb{F}_p$

Integer-Forcing Schemes

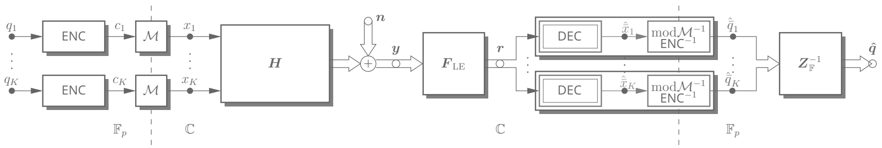
Compute-And-Forward Strategy in Relaying: [NG'11]



- the receiver decodes an *integer linear combination* of the codewords
- resolution of linear combinations at some central unit
  only finite-field symbols are communicated — processing over $\mathbb{F}_p$
- if a joint/central receiver is present, some preprocessing can be done prior to channel decoding — *integer-forcing receiver* [ZNEG'14]

Integer-Forcing Schemes (II)

Integer-Forcing Equalization: [ZNEG'14]



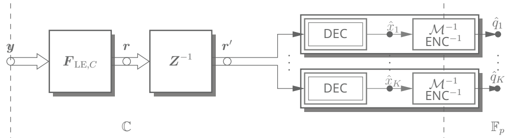
- the users have to use the same *linear code* (or subcodes thereof)
  any integer linear combination of valid codewords is a valid codeword over $\mathbb{F}_p$
- a *linear mapping* has to be applied
  the arithmetics over $\mathbb{F}_p$ has to match that over $\mathbb{R}$ (or $\mathbb{C}$) modulo p [FSK'13]
- this only works if the cardinality of the signal constellation is a *prime number* and equal to the field size p
- the integer matrix has only to be invertible over $\mathbb{F}_p$
  ⇒ *$\boldsymbol{Z}_{\mathbb{F}}$ only has to have full rank*

Equalization Schemes

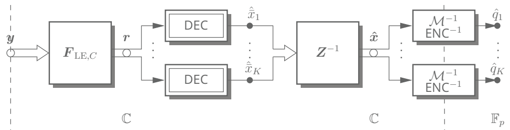
Points to discuss:

- structure
  - LRA vs. IF
  - respective constraints on signal constellations and codes
- factorization task $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{Z}$
  - optimization criterion
  - performance measure
  - suited algorithm
- constraints on $\boldsymbol{Z}$
  - unimodular matrix — $|\det(\boldsymbol{Z})| = 1$
    shortest basis problem
  - full-rank matrix — $\operatorname{rank}(\boldsymbol{Z}) = K$
    shortest independent vector problem

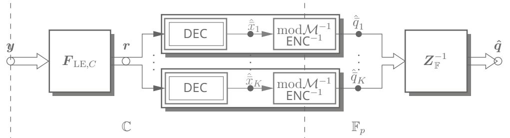
Linear Equalization:



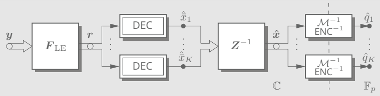
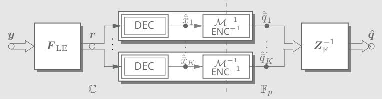
Lattice-Reduction-Aided Equalization: [YW'02], [WF'03]



Integer-Forcing Equalization: [ZNEG'14]



Structure

Lattice-Reduction-Aided Equalization	Integer-Forcing Equalization
	
denomination	
channel-oriented	signal-oriented
suited for	
joint receiver	distributed antenna systems
treat integer interference over	
$\mathbb{G} = \mathbb{Z} + \mathrm{j}\mathbb{Z}$	$\mathbb{F}_p$
constraint on signal constellation and mapping	
usually treated uncoded	incorporation of coding
signal points drawn from a lattice	match arithmetic in $\mathbb{R}$ (or $\mathbb{C}$) and $\mathbb{F}_p$
linear codes over $\mathbb{R}$ (or $\mathbb{C}$)	one-dim. p-ary constellation, p a prime

Factorization Task

Basic Idea of LRA Schemes: [YW'02], [WF'03]

- choose a "more suited" representation of the lattice, a ***reduced basis***
- perform equalization with respect to this new basis;
  integer linear combinations of the data symbols are detected

Procedure:

- input/output relation

$$\boldsymbol{y} \;=\; \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \;=\; \boldsymbol{C}\boldsymbol{Z}\boldsymbol{x} + \boldsymbol{n}$$

- ZF linear equalization of $\boldsymbol{C}$ — equalization matrix $\boldsymbol{F}_{\mathrm{LE},C} = \begin{bmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_K \end{bmatrix} = \boldsymbol{C}^{+}$

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{F}_{\mathrm{LE},C}\boldsymbol{y} \;=\; \boldsymbol{F}_{\mathrm{LE},C}(\boldsymbol{C}\boldsymbol{Z}\boldsymbol{x} + \boldsymbol{n}) \\ &= \boldsymbol{Z}\boldsymbol{x} + \boldsymbol{F}_{\mathrm{LE},C}\boldsymbol{n} \end{aligned}$$

- the noise power in branch k is given by ($\boldsymbol{n}$: i.i.d. components with variance σ_n^2)

$$\sigma_{n_k}^2 \;=\; \sigma_n^2 \cdot \|\boldsymbol{f}_k\|^2 \;=\; \sigma_n^2 \cdot E_k$$

with noise enhancement $E_k = \|\boldsymbol{f}_k\|^2$

Factorization Task (II)

Problem: given $\boldsymbol{H}$, find $\boldsymbol{C}$ and $\boldsymbol{Z}$ such that

- factorization of $\boldsymbol{H}$

$$\boldsymbol{H} = \boldsymbol{C}\boldsymbol{Z}$$

- $\boldsymbol{Z}$ is an *integer matrix*

$$\boldsymbol{Z} \in \mathbb{G}^{K \times K}\,, \qquad \begin{array}{l} \mathrm{rank}(\boldsymbol{Z}) = K \\ \text{if applicable: } |\det(\boldsymbol{Z})| = 1 \quad \text{(unimodular)} \end{array}$$

- $\boldsymbol{C}$, the *"reduced channel"*, or $\boldsymbol{F}_{\mathrm{LE},C}$, the *"equalization matrix"*, have desired properties

Required: to solve this factorization problem, we need

- a meaningful criterion
- a practical algorithm

Factorization Criteria

Criterion I: [YW'02], [WF'03]

- lattice reduction may directly applied to the channel matrix $\boldsymbol{H}$

$$\boldsymbol{H} = \boldsymbol{C}_{\mathrm{I}}\boldsymbol{Z}_{\mathrm{I}}$$

- typically, the *orthogonality defect* of $\boldsymbol{C}_{\mathrm{I}} = \left[\, \boldsymbol{c}_1 \ \cdots \ \boldsymbol{c}_K \,\right]$ is minimized

$$\delta(\boldsymbol{C}_{\mathrm{I}}) = \frac{\prod_{k=1}^{K} \|\boldsymbol{c}_k\|}{|\det(\boldsymbol{C}_{\mathrm{I}})|}$$

- this means that the basis vectors $\boldsymbol{c}_k$, the *column vectors* of $\boldsymbol{C}_{\mathrm{I}}$ should be as short as possible (have small Euclidean norm)

  ⇒ ***shortest basis/independent vector problem***

- a substitute criterion is optimized, instead of system performance

Factorization Criteria (II)

Criterion II: [TMK'07]

- for square channel matrices, the ZF equalization matrix reads

$$\boldsymbol{F}_{\mathrm{LE}} = \boldsymbol{C}^{-1} = \left(\boldsymbol{H}\boldsymbol{Z}^{-1}\right)^{-1} = \boldsymbol{Z}\boldsymbol{H}^{-1}$$

- the squared *row norms* of $\boldsymbol{F}_{\mathrm{LE}}$ give the noise enhancement
- factorization task $\quad (\boldsymbol{X}^{-\mathsf{H}} = (\boldsymbol{X}^{\mathsf{H}})^{-1} = (\boldsymbol{X}^{-1})^{\mathsf{H}})$

$$\boldsymbol{H}^{-\mathsf{H}} = \boldsymbol{F}_{\mathrm{II}}^{\mathsf{H}}\boldsymbol{Z}_{\mathrm{II}}^{-\mathsf{H}}$$

- the *column vectors* of $\boldsymbol{F}_{\mathrm{II}}^{\mathsf{H}}$ should be as short as possible
- if $\boldsymbol{Z}_{\mathrm{II}}$ is an unimodular integer matrix, $\boldsymbol{Z}_{\mathrm{II}}^{-\mathsf{H}}$ has also this property
- for non-square channel matrices the left pseudoinverse is used

$$\left(\boldsymbol{H}^{+}\right)^{\mathsf{H}} = \boldsymbol{F}_{\mathrm{II}}^{\mathsf{H}}\boldsymbol{Z}_{\mathrm{II}}^{-\mathsf{H}}$$

$(\boldsymbol{H} \in \mathbb{C}^{N \times K},\, N \geq K)$

Factorization Criteria (III)

Criterion III: [WBKK'04]

- the MMSE solution can be calculated as ZF solution for the *augmented* channel matrix [Has'00]
- factorization task $(\zeta = \sigma_n^2/\sigma_x^2)$

$$\begin{bmatrix} \boldsymbol{H} \\ \sqrt{\zeta}\boldsymbol{I} \end{bmatrix} \overset{\text{def}}{=} \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{C}}_{\mathrm{III}}\boldsymbol{Z}_{\mathrm{III}} = \begin{bmatrix} \boldsymbol{C}_{\mathrm{III}} \\ \sqrt{\zeta}\boldsymbol{Z}_{\mathrm{III}}^{-1} \end{bmatrix} \boldsymbol{Z}_{\mathrm{III}}$$

- optimum MMSE equalization matrix

$$\begin{aligned} \boldsymbol{F}_{\mathrm{LE,MMSE},C} &= \left[\left(\boldsymbol{\mathcal{C}}_{\mathrm{III}}^{\mathsf{H}}\boldsymbol{\mathcal{C}}_{\mathrm{III}}\right)^{-1}\boldsymbol{\mathcal{C}}_{\mathrm{III}}^{\mathsf{H}}\right]_{\text{left } K \text{ columns}} \\ &= \left(\boldsymbol{C}_{\mathrm{III}}^{\mathsf{H}}\boldsymbol{C}_{\mathrm{III}} + \zeta\boldsymbol{Z}_{\mathrm{III}}^{-\mathsf{H}}\boldsymbol{Z}_{\mathrm{III}}^{-1}\right)^{-1}\boldsymbol{C}_{\mathrm{III}}^{\mathsf{H}} \\ &= \boldsymbol{Z}_{\mathrm{III}}\left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1}\boldsymbol{H}^{\mathsf{H}} = \boldsymbol{Z}_{\mathrm{III}}\boldsymbol{F}_{\mathrm{LE,MMSE},H} \end{aligned}$$

- the *column vectors* of $\boldsymbol{\mathcal{C}}_{\mathrm{III}}$ should be as short as possible
- as in Criterion I, a substitute measure is optimized
- in almost all cases $\boldsymbol{Z}_{\mathrm{I}} = \boldsymbol{Z}_{\mathrm{III}}$ [Fis'11]

Factorization Criteria (IV)

Criterion IV: [FWSSSA'12], [ZNEG'14], [FCS'16]

- applying MMSE linear equalization, the noise enhancement is given by

$$\begin{aligned} E_k &= \left[\boldsymbol{\Phi}_{ee}\right]_{k,k}/\sigma_n^2 = \left[\left(\boldsymbol{C}^{\mathsf{H}}\boldsymbol{C} + \zeta \boldsymbol{Z}^{-\mathsf{H}}\boldsymbol{Z}^{-1}\right)^{-1}\right]_{k,k} \\ &= \left[\boldsymbol{Z}\left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta \boldsymbol{I}\right)^{-1}\boldsymbol{Z}^{\mathsf{H}}\right]_{k,k} = \boldsymbol{z}_k^{\mathsf{H}}\left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1}\boldsymbol{z}_k \\ &= \boldsymbol{z}_k^{\mathsf{H}}\boldsymbol{L}\boldsymbol{L}^{\mathsf{H}}\boldsymbol{z}_k = \|\boldsymbol{L}^{\mathsf{H}}\boldsymbol{z}_k\|^2 \end{aligned}$$

  with $\boldsymbol{Z}^{\mathsf{H}} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]$

- $\boldsymbol{L}$ is any *square root* of $\left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \zeta\boldsymbol{I}\right)^{-1} = \left(\boldsymbol{\mathcal{H}}^{\mathsf{H}}\boldsymbol{\mathcal{H}}\right)^{-1}$; we may choose

$$\boldsymbol{L} = \boldsymbol{\mathcal{H}}^{+}$$

- factorization task (using $\boldsymbol{L}^{\mathsf{H}}\boldsymbol{Z}^{\mathsf{H}} = (\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}\boldsymbol{Z}^{\mathsf{H}} \stackrel{\text{def}}{=} \boldsymbol{\mathcal{F}}^{\mathsf{H}}$)

$$\left(\boldsymbol{\mathcal{H}}^{+}\right)^{\mathsf{H}} = \boldsymbol{\mathcal{F}}_{\mathrm{IV}}^{\mathsf{H}}\,\boldsymbol{Z}_{\mathrm{IV}}^{-\mathsf{H}}$$

- the *column vectors* of $\boldsymbol{\mathcal{F}}_{\mathrm{IV}}^{\mathsf{H}}$ should be as short as possible

Factorization Criteria (V)

Summary: (in each case $\boldsymbol{Z} \in \mathbb{G}^{K\times K}$)

- the criteria available in the literature can be classified as follows

based on	channel matrix $\boldsymbol{H}$ ("ZF solution")	augmented matrix $\boldsymbol{\mathcal{H}}$ ("MMSE solution")
$\boldsymbol{H}$	$\boldsymbol{H} = \boldsymbol{C}\,\boldsymbol{Z}$ [YW'02], [WF'03]	$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{C}}\,\boldsymbol{Z}$ [WBKK'04], [Fis'11]
$(\boldsymbol{H}^{+})^{\mathsf{H}}$	$(\boldsymbol{H}^{+})^{\mathsf{H}} = \boldsymbol{F}^{\mathsf{H}}\,\boldsymbol{Z}^{-\mathsf{H}}$ [TMK'07]	$(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}\,\boldsymbol{Z}^{-\mathsf{H}}$ [ZNEG'14], [FCS'16]

Involved lattices:

$\boldsymbol{H}$: lattice spanned by channel matrix

$(\boldsymbol{H}^{+})^{\mathsf{H}}$: dual lattice [LMG'09]

Constraint on Z

Constraint on the Integer Matrix $\boldsymbol{Z} \in \mathbb{G}^{K\times K}$:

- typically, in LRA equalization it has been forced

$$|\det(\boldsymbol{Z})| = 1 \qquad \text{unimodular matrix}$$

  hence a *change of basis* is performed

  ⇒ ***Lattice Basis Reduction***

- in IF equalization, the constraint is relaxed to

$$\operatorname{rank}(\boldsymbol{Z}) = K \qquad \text{full-rank matrix}$$

  (to be precise: $\operatorname{rank}(\boldsymbol{Z}_{\mathbb{F}}) = K$)

  ⇒ ***Shortest Independent Vector Problem***

Observation: [FCS'16]

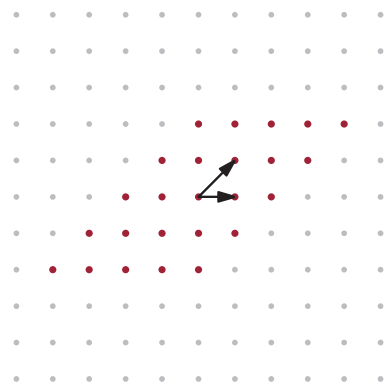
using the LRA equalization structure, unimodularity of $\boldsymbol{Z}$ is not required

⇒ ***both, LRA and IF, can use the same factorization criterion and the same constraint on Z!***

Constraint on Z

Visualization: (real-valued example $K = 2$, $|\mathcal{A}| = 5$)

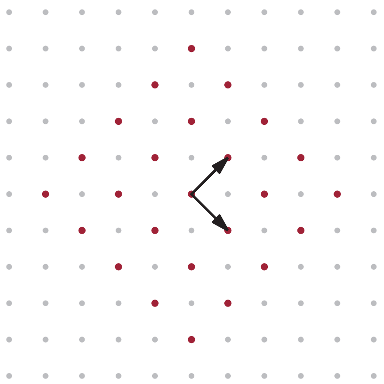
- vectors $\bar{\boldsymbol{x}} = \boldsymbol{Z}\boldsymbol{x}$, with $\boldsymbol{x} \in \mathcal{A}^K$
- example $\quad \boldsymbol{Z} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \det(\boldsymbol{Z}) = 1$



Constraint on Z

Visualization: (real-valued example $K = 2$, $|\mathcal{A}| = 5$)

- vectors $\bar{\boldsymbol{x}} = \boldsymbol{Z}\boldsymbol{x}$, with $\boldsymbol{x} \in \mathcal{A}^K$
- example $\quad \boldsymbol{Z} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \det(\boldsymbol{Z}) = 2$



Lattices and Lattice Problems

Lattice:

- we deal with complex-valued lattices

$$\boldsymbol{\Lambda}(\boldsymbol{G}) = \left\{ \boldsymbol{\lambda} = \sum_{k=1}^{K} z_k \boldsymbol{g}_k = \boldsymbol{G} \begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix} \mid z_k \in \mathbb{G} \right\} \stackrel{\text{def}}{=} \boldsymbol{G}\mathbb{G}^K$$

where

$$\boldsymbol{G} = \left[\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K \right] \in \mathbb{C}^{N \times K}$$

is its generator matrix (basis) consisting of
$K \in \mathbb{N}$ linearly independent basis vectors $\boldsymbol{g}_k \in \mathbb{C}^N$, $N \geq K$, $N \in \mathbb{N}$
(N-dimensional lattice of rank K)

Alternative Description:

- instead of dealing with the complex-valued generator matrix $\boldsymbol{G}$, one can use the real-valued equivalent [Win'04]

$$\boldsymbol{G}_{\text{real}} \stackrel{\text{def}}{=} \begin{bmatrix} \text{Re}\{\boldsymbol{G}\} & -\text{Im}\{\boldsymbol{G}\} \\ \text{Im}\{\boldsymbol{G}\} & \text{Re}\{\boldsymbol{G}\} \end{bmatrix}$$

of doubled dimension

Lattices and Lattice Problems (II)

Gram–Schmidt (GS) Orthogonalization: [Fis'10]

- any matrix $\boldsymbol{G} \in \mathbb{C}^{N \times K}$ can be decomposed into the form

$$\boldsymbol{G} = \boldsymbol{G}^{\circ}\boldsymbol{R}$$

with
- $\boldsymbol{G}^{\circ} = [\boldsymbol{g}_1^{\circ}, \ldots, \boldsymbol{g}_K^{\circ}]$: Gram–Schmidt orthogonalization of $\boldsymbol{G}$ with orthogonal columns $\boldsymbol{g}_1^{\circ}, \ldots, \boldsymbol{g}_K^{\circ}$
- $\boldsymbol{R} = \left[r_{l,k} \right] \in \mathbb{C}^{K \times K}$: upper triangular with unit main diagonal

- successive procedure

for $k = 1, \ldots, K$

$$\boldsymbol{g}_k^{\circ} = \boldsymbol{g}_k - \sum_{l=1}^{k-1} r_{l,k}\, \boldsymbol{g}_l^{\circ}$$

with $\quad r_{l,k} = \dfrac{(\boldsymbol{g}_l^{\circ})^{\mathsf{H}} \boldsymbol{g}_k^{\circ}}{\|\boldsymbol{g}_l^{\circ}\|_2^2}, \qquad l = 1, \ldots, k$

Lattices and Lattice Problems (III)

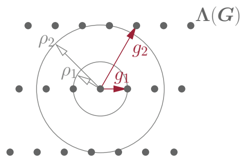
Minkowski's Successive Minima:

- k^{th}, $k = 1, \ldots, K$, *successive minimum* of $\boldsymbol{\Lambda}(\boldsymbol{G})$ [Cas'97], [LLS'90], [DKWZ'15]

$$\rho_k(\boldsymbol{\Lambda}(\boldsymbol{G})) = \inf \left\{ r_k \mid \dim\left(\text{span}\left(\boldsymbol{\Lambda}(\boldsymbol{G}) \cap \boldsymbol{B}_N(r_k)\right)\right) = k \right\}$$

with
- $\boldsymbol{B}_N(r)$: N-dimensional ball (over $\mathbb{C}$) with radius r centered at the origin
- $\text{span}(\cdot)$: linear span

- $\rho_1(\boldsymbol{\Lambda}(\boldsymbol{G}))$ is the norm of the shortest vector of the lattice $\boldsymbol{\Lambda}(\boldsymbol{G})$
- interpretation:
r_k has to be chosen as the smallest radius such that $\boldsymbol{B}_N(r_k)$ contains k linearly independent lattice vectors
- Visualization:



Lattices and Lattice Problems (IV)

Given: a complex-valued lattice $\boldsymbol{\Lambda}(\boldsymbol{G})$ of rank K

Shortest Independent Vector Problem (SIVP):

- find set $\mathcal{G} = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K\}$ of K linearly independent vectors $\boldsymbol{\lambda}_k \in \boldsymbol{\Lambda}(\boldsymbol{G})$, such that
$$\max_{k=1,\dots,K} \|\boldsymbol{\lambda}_k\| \;=\; \rho_K(\boldsymbol{\Lambda}(\boldsymbol{G}))$$
- the largest vector has to be as short as possible; the norms of all shorter vectors do not matter

Successive Minima Problem (SMP):

- find set $\mathcal{G} = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K\}$ of K linearly independent vectors $\boldsymbol{\lambda}_k \in \boldsymbol{\Lambda}(\boldsymbol{G})$, such that
$$\|\boldsymbol{\lambda}_k\| \;=\; \rho_k(\boldsymbol{\Lambda}(\boldsymbol{G}))\,, \qquad k = 1, \dots, K$$
- all lattice vectors in the set $\mathcal{G}$ have to be as short as possible; naturally, SMP is also a solution to SIVP
- efficient strategies for solving the (C)SMP are available [DKWZ'15], [FCS'16]

Lattices and Lattice Problems (V)

Set of Linearly Independent Vectors:

- the obtained vectors are lattice points $\boldsymbol{\lambda}_k \in \boldsymbol{\Lambda}(\boldsymbol{G})$, hence
$$\boldsymbol{\lambda}_k = \boldsymbol{G}\boldsymbol{u}_k\,, \qquad \text{with} \quad \boldsymbol{u}_k \in \mathbb{G}^K, \quad \forall k$$
- the matrix $\boldsymbol{V} \stackrel{\text{def}}{=} [\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K]$ is related to $\boldsymbol{G}$ via
$$\boldsymbol{V} = \boldsymbol{G}\boldsymbol{U}$$
or
$$\boldsymbol{G} = \boldsymbol{V}\boldsymbol{U}^{-1}$$
with $\boldsymbol{U} \in \mathbb{G}^{K\times K}$ and $|\det(\boldsymbol{U})| \in \mathbb{G} \setminus \{0\}$

(cf. factorization task $(\boldsymbol{H}^+)^{\mathsf{H}} = \boldsymbol{F}^{\mathsf{H}}\boldsymbol{Z}^{-\mathsf{H}}$)

Lattices and Lattice Problems (VI)

Lattice Basis Reduction:

- find set $\mathcal{G} = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K\}$ of K linearly independent vectors $\boldsymbol{\lambda}_k \in \boldsymbol{\Lambda}(\boldsymbol{G})$, such that
$$\boldsymbol{\Lambda}(\boldsymbol{G}) \;=\; \boldsymbol{\Lambda}(\boldsymbol{G}_{\mathrm{r}})$$
with
$$\boldsymbol{G}_{\mathrm{r}} \;=\; [\boldsymbol{g}_{\mathrm{r},1}, \dots, \boldsymbol{g}_{\mathrm{r},K}] \;=\; [\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K]$$
i.e., $\boldsymbol{G}_{\mathrm{r}}$ is a "reduced" basis of the lattice $\boldsymbol{\Lambda}$
(the meaning of "reduced" depends on the criterion/algorithm)
- the generator matrices are related by
$$\boldsymbol{G}_{\mathrm{r}} = \boldsymbol{G}\boldsymbol{U}$$
or
$$\boldsymbol{G} = \boldsymbol{G}_{\mathrm{r}}\boldsymbol{U}^{-1}$$
where $\boldsymbol{U} \in \mathbb{G}^{K\times K}$ is unimodular, i.e., $|\det(\boldsymbol{U})| = 1$;
hence $\boldsymbol{U}^{-1} \in \mathbb{G}^{K\times K}$

(cf. factorization task $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{Z}$)

Lattices and Lattice Problems (VII)

Lenstra-Lenstra-Lovász (LLL) Reduction: [LLL'82]

- a generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \dots, \boldsymbol{g}_K] \in \mathbb{C}^{N\times K}$ with Gram–Schmidt orthogonal basis $\boldsymbol{G}^\circ = [\boldsymbol{g}_1^\circ, \dots, \boldsymbol{g}_K^\circ]$ and upper triangular matrix $\boldsymbol{R}$ is called (C)LLL-reduced, if [GLM'09]
  1. for $1 \le l < k \le K$, it is *size-reduced* according to
  $$|\mathrm{Re}\{r_{l,k}\}| \le 0.5 \qquad \text{and} \qquad |\mathrm{Im}\{r_{l,k}\}| \le 0.5$$
  2. for $k = 2, \dots, K$ and a parameter $0.5 < \delta \le 1$
  $$\|\boldsymbol{g}_k^\circ\|^2 \ge (\delta - |r_{k-1,k}|^2)\|\boldsymbol{g}_{k-1}^\circ\|^2$$
- the parameter δ controls the trade-off between "strength" of the LLL reduction and computational complexity — usually $\delta = 0.75$; the case $\delta = 1$ is denoted as *optimal LLL reduction* [A'03]
- for $\delta < 1$ the algorithm has polynomial complexity [A'03]

Lattices and Lattice Problems (VIII)

Hermite-Korkine-Zolotareff (HKZ) Reduction:

- a generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K] \in \mathbb{C}^{N\times K}$ with Gram–Schmidt orthogonal basis $\boldsymbol{G}^\circ = [\boldsymbol{g}_1^\circ, \ldots, \boldsymbol{g}_K^\circ]$ and upper triangular matrix $\boldsymbol{R}$ is called (C)HKZ-reduced, if [LLS'90], [JD'13]

  1. for $1 \le l < k \le K$, it is *size-reduced* according to
  $$|\mathrm{Re}\{r_{l,k}\}| \le 0.5 \quad \text{and} \quad |\mathrm{Im}\{r_{l,k}\}| \le 0.5$$
  2. for $k = 1, \ldots, K$, the columns of $\boldsymbol{G}^\circ$ fulfill
  $$\|\boldsymbol{g}_k^\circ\| = \rho_1(\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)}))$$
  (shortest (non-zero) vector in $\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$)

- $\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$: sublattice of rank $K-k+1$ and dimension N with generator matrix $\boldsymbol{G}^{(k)} = [0, \ldots, 0, \boldsymbol{g}_k^\circ, \ldots, \boldsymbol{g}_K^\circ]\boldsymbol{R}$
  ($\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$ is the orth. projection of $\boldsymbol{\Lambda}(\boldsymbol{G})$ onto the orth. complement of $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{k-1}\}$)

- since shortest vectors have to be found, the problem is NP-hard; efficient (complex-valued) algorithms available [JD'13], [ZQW'12]

Lattices and Lattice Problems (IX)

Minkowski (MK) Reduction:

- a generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K] \in \mathbb{C}^{N\times K}$ is called (C)MK-reduced, if [Min'1891], [ZQW'12]
$$\begin{aligned} \|\boldsymbol{g}_k\| &\le \|\boldsymbol{g}'_k\|\,, \quad k = 1, \ldots, K \\ \forall \boldsymbol{G}' &= [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}'_k, \ldots, \boldsymbol{g}'_K] \\ \text{with} \quad \boldsymbol{\Lambda}(\boldsymbol{G}') &= \boldsymbol{\Lambda}(\boldsymbol{G}) \end{aligned}$$
  $\boldsymbol{G}$ is Minkowski-reduced if for $k = 1, \ldots, K$ the basis vector $\boldsymbol{g}_k$ has minimum norm among all possible lattice points $\boldsymbol{g}'_k$ for which the set $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}'_k\}$ can be extended to a basis of $\boldsymbol{\Lambda}(\boldsymbol{G})$

- in contrast to the SMP where only the K shortest *independent* lattice vectors have to be found, here the K shortest vectors have to be obtained that *form a basis of the lattice*

- efficient (real-valued) algorithm available [ZQW'12]
  in the real-valued case, the calculation of a greatest common divisor (gcd) is required; in the complex-valued case the gcd for Gaussian integers has to be used (calculated via the Euclidean Algorithm)

Application to Equalization

Recall: Criterion IV

- MMSE linear equalization via $\boldsymbol{\mathcal{F}}^{\mathsf{H}} = \boldsymbol{Z}\boldsymbol{\mathcal{H}}^{+} = \begin{bmatrix} \mathfrak{f}_1 \\ \vdots \\ \mathfrak{f}_K \end{bmatrix}$
- noise enhancement
$$E_k = \|\,\mathfrak{f}_k\,\|^2 = \|(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}\boldsymbol{z}_k\|^2 \to \min$$
  with $\boldsymbol{Z}^{\mathsf{H}} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]$
- factorization task
$$(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}\,\boldsymbol{Z}^{-\mathsf{H}}$$
- the *column vectors* of $\boldsymbol{\mathcal{F}}^{\mathsf{H}}$ should be as short as possible
- usually the maximum of the noise enhancement dominates

Application to Equalization (II)

Factorization Problem: $\boldsymbol{Z}^{\mathsf{H}} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]$

- $|\det(\boldsymbol{Z}^{\mathsf{H}})| = 1$ required
$$\boldsymbol{Z}^{\mathsf{H}} = \underset{\substack{\boldsymbol{Z}^{\mathsf{H}} \in \mathbb{G}^{K\times K} \\ |\det(\boldsymbol{Z}^{\mathsf{H}})|=1}}{\operatorname{argmin}} \; \max_{k=1,\ldots,K} \left\|(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}\boldsymbol{z}_k\right\|^2$$
  ⇒ *shortest basis problem (SBP)*
- the MK-reduced basis is directly defined by the length of its basis vectors — it consists of the K shortest lattice vectors that form a basis of the lattice (not only the maximum norm is minimized)
  ⇒ ***Minkowski reduction gives the optimum integer matrix $\boldsymbol{Z}$***
- full-rank matrix $\boldsymbol{Z}$ sufficient
$$\boldsymbol{Z}^{\mathsf{H}} = \underset{\substack{\boldsymbol{Z}^{\mathsf{H}} \in \mathbb{G}^{K\times K} \\ \mathrm{rank}(\boldsymbol{Z}^{\mathsf{H}})=K}}{\operatorname{argmin}} \; \max_{k=1,\ldots,K} \left\|(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}\boldsymbol{z}_k\right\|^2$$
  ⇒ *shortest independent vector problem (SIVP)*
- this problem is optimally solved—in a stricter sense—if the K successive minima of $\boldsymbol{\Lambda}((\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}})$ are obtained
  ⇒ ***Minkowski's successive minima give the optimum integer matrix $\boldsymbol{Z}$***

Numerical Results

Obtained Vectors z_k:

- factorization of $\boldsymbol{G} = \begin{bmatrix} 0.8+0.5\mathrm{j} & -0.8+0.1\mathrm{j} & -0.1-0.6\mathrm{j} & 0.7-1.0\mathrm{j} \\ -0.5+0.4\mathrm{j} & -0.1-0.2\mathrm{j} & -1.1+0.8\mathrm{j} & -0.3-1.0\mathrm{j} \\ 0.3-0.5\mathrm{j} & 1.1+2.1\mathrm{j} & 0.8-0.3\mathrm{j} & 0.4+1.4\mathrm{j} \\ -0.3-0.2\mathrm{j} & -1.0+0.0\mathrm{j} & 0.6-0.4\mathrm{j} & 0.2+1.1\mathrm{j} \end{bmatrix}$

$\boldsymbol{u}_i$	$\begin{bmatrix}3+0\mathrm{j}\\0+1\mathrm{j}\\-2-1\mathrm{j}\\1-2\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}2+0\mathrm{j}\\0+1\mathrm{j}\\-2-1\mathrm{j}\\1-2\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}1+0\mathrm{j}\\0+0\mathrm{j}\\-1-1\mathrm{j}\\1-1\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}1+1\mathrm{j}\\0+0\mathrm{j}\\0-1\mathrm{j}\\1+0\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}1+0\mathrm{j}\\0+0\mathrm{j}\\-1+0\mathrm{j}\\0-1\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}3+0\mathrm{j}\\0+1\mathrm{j}\\-3-1\mathrm{j}\\1-3\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}4-1\mathrm{j}\\0+1\mathrm{j}\\-5-1\mathrm{j}\\1-4\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}4+0\mathrm{j}\\0+1\mathrm{j}\\-4-1\mathrm{j}\\1-3\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}3+0\mathrm{j}\\0+1\mathrm{j}\\-4-1\mathrm{j}\\1-3\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}1+0\mathrm{j}\\0+0\mathrm{j}\\0-1\mathrm{j}\\1+0\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0+0\mathrm{j}\\0+0\mathrm{j}\\0-1\mathrm{j}\\1+0\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}1+0\mathrm{j}\\0+0\mathrm{j}\\0+0\mathrm{j}\\0+0\mathrm{j}\end{bmatrix}$
$\boldsymbol{\lambda}_i$	$\begin{bmatrix}0.6-0.4\mathrm{j}\\-0.6+0.2\mathrm{j}\\0.1-0.0\mathrm{j}\\-0.1-0.7\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}-0.2-0.9\mathrm{j}\\-0.1-0.2\mathrm{j}\\-0.2+0.5\mathrm{j}\\0.2-0.5\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.0-0.5\mathrm{j}\\0.1+0.0\mathrm{j}\\1.0-0.0\mathrm{j}\\0.0+0.5\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.4+0.4\mathrm{j}\\-0.4+0.0\mathrm{j}\\0.9+0.4\mathrm{j}\\-0.3+0.0\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}-0.1+0.4\mathrm{j}\\-0.4-0.1\mathrm{j}\\0.9-0.6\mathrm{j}\\0.2+0.0\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}-0.3-0.5\mathrm{j}\\-0.5-0.3\mathrm{j}\\0.7-0.1\mathrm{j}\\0.4-0.5\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.2-0.3\mathrm{j}\\0.6-0.7\mathrm{j}\\0.3-0.7\mathrm{j}\\-0.2+0.2\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.6+0.6\mathrm{j}\\0.1-0.7\mathrm{j}\\0.2-0.3\mathrm{j}\\-0.5-0.3\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}-0.2+0.1\mathrm{j}\\0.6-1.1\mathrm{j}\\-0.1+0.2\mathrm{j}\\-0.2-0.1\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.9-0.4\mathrm{j}\\0.0+0.5\mathrm{j}\\0.4+0.1\mathrm{j}\\-0.5+0.3\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.1-0.9\mathrm{j}\\0.5+0.1\mathrm{j}\\0.1+0.6\mathrm{j}\\-0.2+0.5\mathrm{j}\end{bmatrix}$	$\begin{bmatrix}0.8+0.5\mathrm{j}\\-0.5+0.4\mathrm{j}\\0.3-0.5\mathrm{j}\\-0.3-0.2\mathrm{j}\end{bmatrix}$
$\Vert\boldsymbol{\lambda}_i\Vert^2$	1.43	1.48	1.51	1.54	1.55	1.59	1.64	1.69	1.72	1.73	1.74	1.77
rank	1	2	3	3	3	3	4	4	4	4	4	4
LLL $_{\delta=.75}$	X			X				X				X
LLL $_{\delta=1}$			X	X		X	X					
HKZ	X	X						X		X		
MK	X	X		X			X					
SMP	X	X	X				X					

- here: $\det(\boldsymbol{Z}_{\text{SMP}}) = 1 + \mathrm{j}$

Numerical Results (II)

Distribution of $|\det(\boldsymbol{Z})|$:

- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian; $K = N$
- $\sigma_x^2/\sigma_n^2 \mathrel{\hat{=}} 20$ dB
- criterion IV — SMP [DKWZ'15], [FCS'16]

$\vert\det(\boldsymbol{Z})\vert =$	1	$\sqrt{2}$	2	$\sqrt{5}$
$K = 2$	100 %	—	—	—
$K = 3$	99.8 %	0.2 %	—	—
$K = 4$	99.0 %	1.0 %	—	—
$K = 5$	97.5 %	2.4 %	0.005 %	—
$K = 6$	95.6 %	4.5 %	0.03 %	0.003 %
$K = 7$	92.7 %	7.1 %	0.15 %	0.02 %
$K = 8$	89.3 %	10.2 %	0.39 %	0.06 %

Numerical Results (III)

Distribution of $|\det(\boldsymbol{Z})|$:

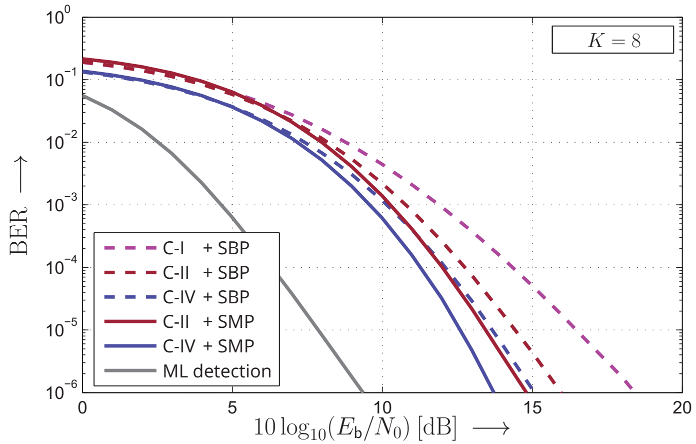
- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian
- $K = N = 6$
- criterion IV — SMP [DKWZ'15], [FCS'16]

$\vert\det(\boldsymbol{Z})\vert =$	1	$\sqrt{2}$	2	$\sqrt{5}$
$\sigma_x^2/\sigma_n^2 \mathrel{\hat{=}} 0$ dB	99.6 %	0.45 %	0.0002 %	—
$\sigma_x^2/\sigma_n^2 \mathrel{\hat{=}} 10$ dB	96.2 %	3.83 %	0.02 %	0.002 %
$\sigma_x^2/\sigma_n^2 \mathrel{\hat{=}} 20$ dB	95.4 %	4.45 %	0.03 %	0.003 %
$\sigma_x^2/\sigma_n^2 \mathrel{\hat{=}} 30$ dB	95.5 %	4.48 %	0.03 %	0.003 %

Numerical Results (IV)

Bit Error Rate: LRA structure; linear MMSE equalization — different criteria and constraints

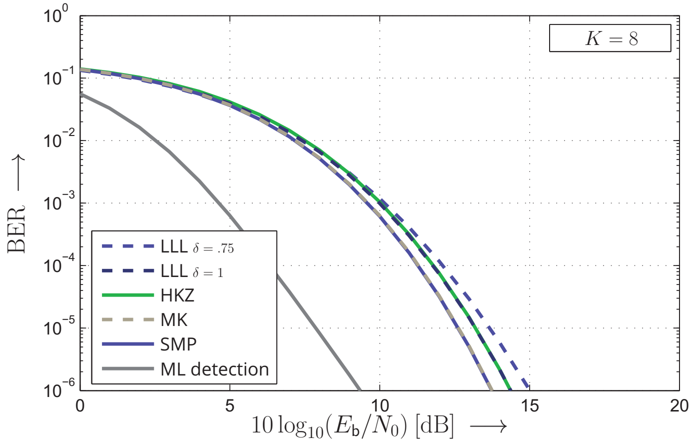
- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian; $K = N$
- uncoded transmission; 16QAM signaling; $E_\mathrm{b}/N_0 = \sigma_x^2/(\sigma_n^2 \log_2(16))$



Numerical Results (V)

Bit Error Rate: LRA structure; linear MMSE equalization; criterion C-IV — different algorithms

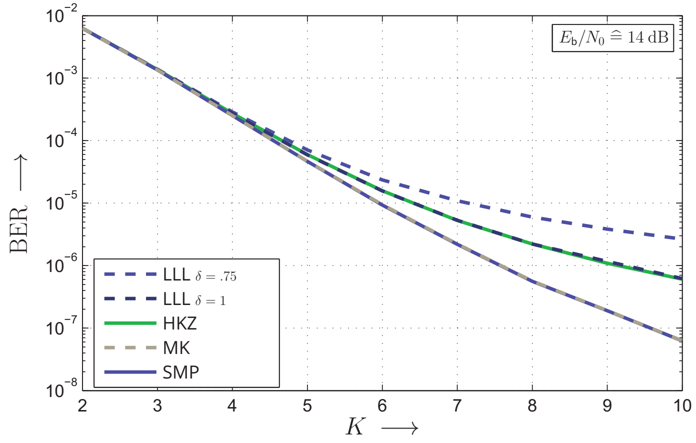
- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian; $K = N$
- uncoded transmission; 16QAM signaling; $E_\mathsf{b}/N_0 = \sigma_x^2/(\sigma_n^2 \log_2(16))$



Numerical Results (VI)

Bit Error Rate: LRA structure; linear MMSE equalization; criterion C-IV — different algorithms

- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian; $K = N$
- uncoded transmission; 16QAM signaling



Numerical Results (VII)

Percentages "MK = SMP" and "MK = SIVP":

- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian
- $K = N$; criterion IV [DKWZ'15], [FCS'16]

SMP | $K = N =$	2	4	6	8	10
$\sigma_x^2/\sigma_n^2 \widehat{=} 15$ dB	100 %	99.0 %	95.7 %	90.3 %	83.8 %
$\sigma_x^2/\sigma_n^2 \widehat{=} 20$ dB	100 %	99.0 %	95.6 %	89.8 %	82.3 %
$\sigma_x^2/\sigma_n^2 \to \infty$	100 %	99.0 %	95.5 %	89.4 %	81.5 %

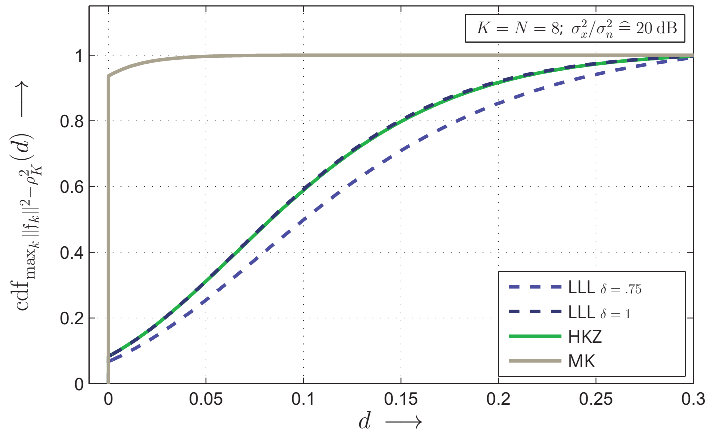
SIVP | $K = N =$	2	4	6	8	10
$\sigma_x^2/\sigma_n^2 \widehat{=} 15$ dB	100 %	99.2 %	97.0 %	94.0 %	90.6 %
$\sigma_x^2/\sigma_n^2 \widehat{=} 20$ dB	100 %	99.2 %	97.0 %	93.5 %	89.3 %
$\sigma_x^2/\sigma_n^2 \to \infty$	100 %	99.2 %	96.9 %	93.2 %	88.5 %

for the complex case and $K = N = 2$, an MK-reduced basis is always a solution to the SMP

Numerical Results (VIII)

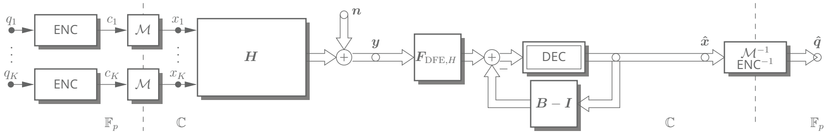
Distribution of Deviation from Optimum:

- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian
- $K = N = 8$; $\sigma_x^2/\sigma_n^2 \widehat{=} 20$ dB
- criterion IV [DKWZ'15], [FCS'16]



Decision-Feedback Equalization

Decision-Feedback Equalization: aka successive interference cancellation, V-BLAST



- QR decomposition of the channel matrix:
  $\boldsymbol{Q}$: orthogonal matrix; $\boldsymbol{B}$: upper triangular, unit main diagonal

$$\boldsymbol{H} = \boldsymbol{Q}\boldsymbol{B}$$

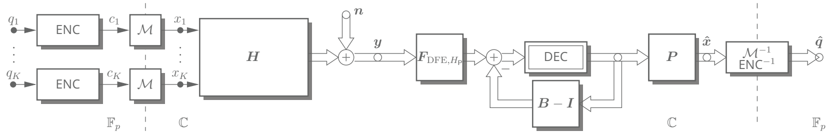
- signal after feedforward processing with $\boldsymbol{F}_{\mathrm{DFE},H} \stackrel{\text{def}}{=} (\boldsymbol{Q}^{\mathsf{H}}\boldsymbol{Q})^{-1}\boldsymbol{Q}^{\mathsf{H}}$

$$\boldsymbol{r} = \boldsymbol{F}_{\mathrm{DFE},H}\boldsymbol{y} = \boldsymbol{B}\boldsymbol{x} + \tilde{\boldsymbol{n}}$$

  - spatially causal signal transmission matrix $\boldsymbol{B}$
  - Gaussian noise vector $\tilde{\boldsymbol{n}}$ with correlation matrix $\sigma_n^2(\boldsymbol{Q}^{\mathsf{H}}\boldsymbol{Q})^{-1}$
    i.e., with $\boldsymbol{Q} = [\,\boldsymbol{q}_1 \cdots \boldsymbol{q}_K\,]$ noise variances $\sigma^2_{\tilde{n}_k} = \sigma_n^2/\|\boldsymbol{q}_k\|^2$
  - decisions are taken successively (order $K, \ldots, 1$)

Decision-Feedback Equalization

Decision-Feedback Equalization: aka successive interference cancellation, V-BLAST



- *sorted* QR decomposition of the channel matrix:
  $\boldsymbol{Q}$: orthogonal matrix; $\boldsymbol{B}$: upper triangular, unit main diagonal; $\boldsymbol{P}$: permutation matrix

$$\boldsymbol{H}\boldsymbol{P} \stackrel{\text{def}}{=} \boldsymbol{H}_{\mathrm{P}} = \boldsymbol{Q}\boldsymbol{B}$$

⇒ criterion for sorting required

MMSE version of DFE:

- ZF version for $K = N$: $\quad \boldsymbol{H}\boldsymbol{P} = \boldsymbol{F}^{-1}\boldsymbol{B}$
- MMSE version of DFE: $\quad \boldsymbol{\mathcal{H}}\boldsymbol{P} = \boldsymbol{\mathcal{F}}^{+}\boldsymbol{B}$
  with $\boldsymbol{\mathcal{H}} = \begin{bmatrix} \boldsymbol{H} \\ \sqrt{\zeta}\boldsymbol{I} \end{bmatrix}$

Decision-Feedback Equalization (II)

Optimum Detection Order: V-BLAST ordering [WFGV'98]

- signal-to-noise ratio in component k is proportional to $\|\boldsymbol{q}_k\|^2$

⇒ for $k = K, \ldots, 1$: the norm of the vector $\boldsymbol{q}_k$ should be the largest among the remaining components $1, \ldots, k$

- BLAST ordering requires great effort

Simpler Strategy: [WBKK'03], [Fis'10]

- instead of *maximizing* $\|\boldsymbol{q}_k\|^2$ in sequence $k = K, K-1, \ldots, 1$
  it is *minimized* in sequence $k = 1, 2, \ldots, K$

⇒ for $k = 1, \ldots, K$: the norm of the vector $\boldsymbol{q}_k$ should be the smallest among the remaining components $k, \ldots, K$

- ***Gram–Schmidt procedure with pivoting***

Simple but Optimum Strategy: [LMG'09]

- do not apply Gram–Schmidt procedure with pivoting to $\boldsymbol{\mathcal{H}}$, but to $(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}$

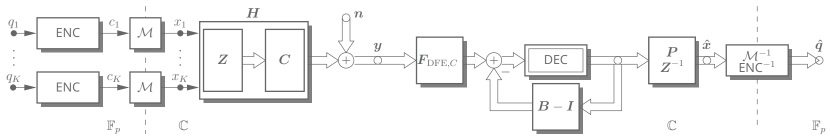
⇒ use factorization

$$(\boldsymbol{\mathcal{H}}^{+})^{\mathsf{H}}\boldsymbol{P}^{-\mathsf{H}} = \boldsymbol{\mathcal{F}}^{\mathsf{H}}\boldsymbol{B}^{-\mathsf{H}}$$

order within GS proc.: $k = K, \ldots, 1$; i.e., $\boldsymbol{B}^{-\mathsf{H}}$ should be lower triangular

LRA Decision-Feedback Equalization

LRA Decision-Feedback Equalization: [YW'02], [WF'03]



Strategies:

- obvious [YW'02], [WF'03]
  perform i) factorization $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{Z}$;
  ii) sorted QR decomposition $\boldsymbol{C}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{B}$
- more efficient [WBKK'04], [Fis'11]
  reuse $\boldsymbol{Q}$ and $\boldsymbol{R}$ anyway calculated within LLL or HKZ
- optimum [LMG'09], [Fis'10], [SF'17]
  do sorting, Gram–Schmidt procedure, and size reduction jointly

LRA Decision-Feedback Equalization (II)

Pseudocode of Factorization Approach: [SF'17]

$[\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}] = \texttt{GramSchmidtSort\_LRA}(\boldsymbol{G})$

$\quad \boldsymbol{Q} = \boldsymbol{G}, \boldsymbol{R} = \boldsymbol{I}, \boldsymbol{T} = \boldsymbol{I}$
$\quad k = 1$
$\quad$ **while** $k \le K$ {
$\qquad \boldsymbol{q}_\mathrm{s} = \texttt{shortest vector in } \boldsymbol{\Lambda}([\boldsymbol{q}_k, \ldots, \boldsymbol{q}_K])$
$\qquad$ **if** $\|\boldsymbol{q}_\mathrm{s}\|^2 \neq \|\boldsymbol{q}_k\|^2$ {
$\qquad\quad \boldsymbol{q}_k = \boldsymbol{q}_\mathrm{s}$
$\qquad\quad \texttt{update } \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T} \texttt{ such that } \boldsymbol{\Lambda}(\boldsymbol{QR}) = \boldsymbol{\Lambda}(\boldsymbol{G})$
$\qquad$ }
$\qquad$ **for** $i = k+1, \ldots, K$ {
$\qquad\quad r_{ki} = \boldsymbol{q}_k^\mathsf{H} \boldsymbol{q}_i / \|\boldsymbol{q}_k\|^2$
$\qquad\quad \boldsymbol{q}_i = \boldsymbol{q}_i - r_{ki} \boldsymbol{q}_k$
$\qquad$ }
$\qquad k = k + 1$
$\quad$ }

LRA Decision-Feedback Equalization (III)

Recall: Hermite-Korkine-Zolotareff (HKZ) Reduction

- a generator matrix $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K] \in \mathbb{C}^{N \times K}$ with Gram–Schmidt orthogonal basis $\boldsymbol{G}^\circ = [\boldsymbol{g}_1^\circ, \ldots, \boldsymbol{g}_K^\circ]$ and upper triangular matrix $\boldsymbol{R}$ is called (C)HKZ-reduced, if [LLS'90], [JD'13]
  1. for $1 \le l < k \le K$, it is *size-reduced* according to
  $$|\mathrm{Re}\{r_{l,k}\}| \le 0.5 \quad \text{and} \quad |\mathrm{Im}\{r_{l,k}\}| \le 0.5$$
  2. for $k = 1, \ldots, K$, the columns of $\boldsymbol{G}^\circ$ fulfill
  $$\|\boldsymbol{g}_k^\circ\| = \rho_1(\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)}))$$
  (shortest (non-zero) vector in $\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$)
- $\boldsymbol{\Lambda}(\boldsymbol{G}^{(k)})$: sublattice with generator matrix $\boldsymbol{G}^{(k)} = [0, \ldots, 0, \boldsymbol{g}_k^\circ, \ldots, \boldsymbol{g}_K^\circ]\boldsymbol{R}$

LRA Decision-Feedback Equalization (IV)

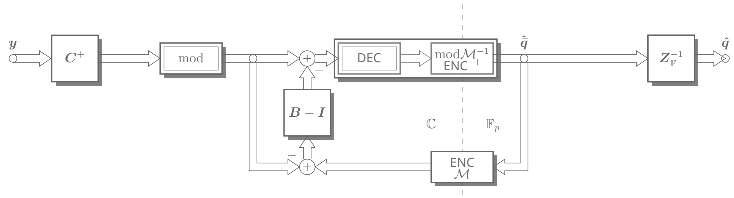
Discussion:

- the size-reduction step of HKZ is not present;
  as it changes only $\boldsymbol{R}$ it is of no relevance for performance of LRA DFE
  ⇒ *effective HKZ reduction*
- for $\boldsymbol{G} = (\boldsymbol{\mathcal{H}}^+)^\mathsf{H}$ the algorithms returns $\boldsymbol{Z}^\mathsf{H} = \boldsymbol{T}$ and $\boldsymbol{\mathcal{F}}^\mathsf{H} = \boldsymbol{Q}$ with
  - V-BLAST sorting
  - the columns of $\boldsymbol{\mathcal{F}}^\mathsf{H}$ have minimum norm
    (optimal worst-link performance as in classical V-BLAST but for LRA equalization)
- this optimum is achieved with an unimodular $\boldsymbol{Z}$;
  a relaxation to $\mathrm{rank}(\boldsymbol{Z}) = K$ is not required [OEN'13]

⇒ ***successive IF and LRA DFE both can be restricted to unimodular $\boldsymbol{Z}$***

LRA Decision-Feedback Equalization (V)

LRA Decision-Feedback Equalization:



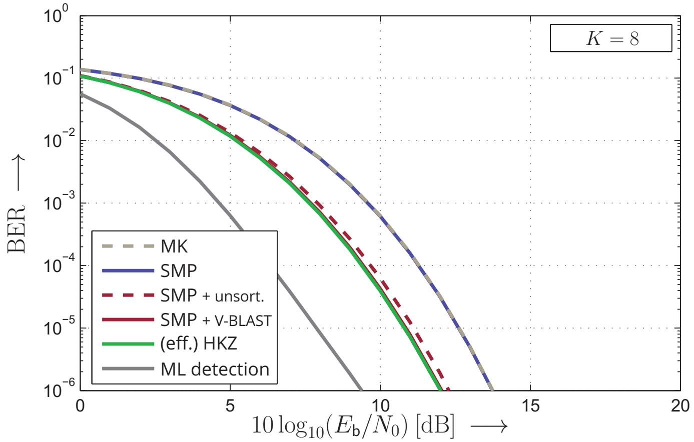
- redraw to noise-prediction structure [Fis'02]
- apply modulo reduction w.r.t. $\boldsymbol{\Lambda}_\mathrm{s}$
- exchange $\boldsymbol{Z}^{-1}$ and demapping/encoder inverse
- combine to demapping modulo $\boldsymbol{\Lambda}_\mathrm{s}$

⇒ ***successive IF only works in noise-prediction structure***

Numerical Results

Bit Error Rate: LRA structure; linear MMSE equalization; criterion C-IV — different algorithms

- $\boldsymbol{H}$: i.i.d. random zero-mean unit-variance complex Gaussian; $K = N$
- uncoded transmission; 16QAM signaling; $E_b/N_0 = \sigma_x^2/(\sigma_n^2 \log_2(16))$



Summary

Low-Complexity Equalization Schemes:

- tight relation between LRA and IF equalization
  - ⇒ *structure how equalization and decoding are combined*
- performance measure for defining the factorization task
  - ⇒ *optimization criterion*
- constraints on the integer matrix — SBP vs. SIVP
  - ⇒ *algorithms for performing the factorization*

Optimum Integer Matrix $\boldsymbol{Z}$:

- linear equalization
  - $|\det(\boldsymbol{Z})| = 1$ *Minkowski reduction gives the optimum*
  - $\mathrm{rank}(\boldsymbol{Z}) = K$ *Minkowski's successive minima give the optimum*
- decision-feedback equalization
  (effective) HKZ reduction gives the optimum
  (relaxation to $|\det(\boldsymbol{Z})| > 1$ not required)

Dualization:

- transmitter-side precoding for broadcast channel
  (LRA / IF precoding) [HC'13], [HNS'14], [SF'15]

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